Lectures on Cyclic Cohomology

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Contents

Introduction

In September 2016 I gave 5 introductory lectures on cyclic cohomology and some of its applications in IMPAN Warsaw, during the Simons Semester in Noncommutative Geometry. The audience consisted of graduate students and postdocs and my task was to introduce them to the subject. The following text is an expanded version of my lectures. In producing these lecture notes I have freely used material from my book, *Basic Noncommutative Geometry.* I would like to thank Piotr Hajac for the invitation to give these lectures.

Lecture 1: Index theory in a noncommutative setting

0.1 Index theory in a noncommutative setting

Lecture 1, September 12, 2016. In this first lecture, we try to answer a fundamental question: *what is index theory and how it relates to noncommutative geometry?* We shall see that the need to extend the Atiyah-Singer index theorem beyond its original domain had much influence on the development of noncommutative geometry and its basic tools like cyclic cohomology and Connes- Chern character maps. We shall then formulate the first index theorem in a noncommutative setting due to Connes [19] that exhibits this point in a very clar way . Most of what we say in these five lectures are geared towards understanding the statement of this noncommutative index theorem and its proof. In the last lecture we shall give some applications of this index theorem.

We shall freely use concepts of functional analysis like Fredholm operators, compact operators and p-summable operators, and notions of differential and algebraic topology, topological K-theory, and differential geometry. The reader can consult the appendix for abstract Fredholm theory and K-theory.

0.2 Elliptic operators

Elliptic differential operators acting on smooth sections of vector bundles over closed manifolds define Fredholm operators on the corresponding Sobolev spaces of sections. Computing the index of such Fredholm operators in terms of topological data is what the index theorem of Atiyah–Singer achieves. Let M be a smooth manifold and let E and F be smooth complex vector bundles on M. Let

$$D\colon C^\infty(E) \to C^\infty(F)$$

be a liner differential operator. This means that D is a $\mathbb{C}$-linear map which is locally expressible by a matrix of differential operators. This matrix of course depends on the choice of local coordinates on M and local frames for E and F. The *principal symbol* of D is defined by replacing differentiation by covectors in the leading order terms of D. The resulting 'matrix-valued function' on the cotangent bundle'

$$\sigma_D \in C^\infty(\mathrm{Hom}(\pi^* E, \pi^* F))$$

can be shown to be invariantly defined. Here $\pi\colon T^*M \to M$ is the natural projection map of the cotangent bundle.

To compute the principal symbol we can proceed as follows. Let $D = [D_{ij}]$, denote the matrix of the differential operators D in some local coordinate system for M, and local frames for E and F. Thus $D_{ij} = \sum_{|k|\le d} a^k_{ij}(x)\partial_k$ with $d =$ degree of D. Then $\sigma_D(x,\xi) = \left[\sigma_{D_{ij}}(x,\xi)\right]$, where

$$\sigma_{D_{ij}}(x,\xi) = \sum_{|k|=d} a^k_{ij}(x)(\frac{1}{i})^{|k|}\xi^k$$

is obtained by replacing the partial derivatives by covectors: $\partial_k \to (\frac{1}{i})^{|k|}\xi^k$.

Definition 0.2.1. A differential operator D is called an *elliptic* operator if for all $x \in M$ and all $\xi \neq 0$ in $T^*_x M$, the matrix $\sigma_D(x,\xi)$ is invertible.

Theorem 0.2.1. *(Finiteness theorem) Let M be a smooth and closed manifold and E and F smooth complex vector bundles over M. If $D : C^\infty(E) \to C^\infty(F)$ is an elliptic operator, then*

$$\dim(\ker D) < \infty \quad \textit{and} \quad \dim(\mathrm{coker}\, D) < \infty$$

In particular, we can define the index of D by

$$\textit{index}(D) = \dim(\ker D) - \dim(\mathrm{coker}\, D).$$

Let $W^s(E)$ denote the Sobolev space of sections of E (roughly speaking, it consists of sections whose 'derivatives of order s' are square integrable). The main results of the theory of linear elliptic PDE's show that for each $s \in \mathbb{R}$, D has a unique extension to a bounded and Fredholm operator $D\colon W^s(E) \to W^{s-d}(F)$. Moreover the Fredholm index of D is independent of s and coincides with the index defined using smooth sections.

We start with a few classic examples of elliptic operators and their indices.

0.3 The Gauss-Bonnet-Chern theorem as an index theorem

Let M be a Riemannian manifold, and let $d : \Omega^p(M) \to \Omega^{p+1}(M)$, where $\Omega^p(M) := C^\infty(\bigwedge^p T^*M)$ is the space of p-forms on M, denote the de Rham differential. It is a differential operator but is not elliptic. Let $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$ denote the adjoint of d. Here $d^* = -*d*$, with $* : \Omega^p(M) \to \Omega^{n-p}(M)$ the Hodge $*$ operator. Consider the differential operator

$$d + d^* : \Omega^{ev}(M) \longrightarrow \Omega^{odd}(M),$$

where $\Omega^{ev}(M) = \oplus\Omega^{2p}(M)$.This is an elliptic operator. In fact it is easy to find the symbol of $d + d^*$:

$$\sigma_{d+d^*}(x, \xi) : \wedge^{ev}(T_x^*M) \to \wedge^{odd}(T_x^*M)$$

is given by

$$\sigma_{d+d^*}(x, \xi) = \sqrt{-1}(\xi \pm i_\xi),$$

where i_ξ is the interior multiplication by ξ. A simple computation then shows that $\sigma_{d+d^*}(x, \xi)$ is invertible for $\xi \neq 0$ and therefore $d + d^*$ is an elliptic operator. Consequently, $index(d + d^*)$ is defined. Using Hodge theory, one shows that

Proposition 0.3.1. *With the notations and assumptions as above*

$$index(d + d^*) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

In fact, with regard to $\ker(d + d^*) : \Omega^{ev} \to \Omega^{odd}$, we have $(d + d^*)w = 0$ if and only if $dw = 0$ and $d^*w = 0$ (we can assume ω is homogeneous). Consider the *Laplacian* on forms

$$\Delta := (d + d^*)^2 = dd^* + d^*d.$$

We have

$$\ker(d + d^*)|_{\Omega^{ev}} = \ker \Delta = \bigoplus_p \mathcal{H}^{2p}(M),$$

where $\mathcal{H}^{2p}(M)$ is the space of *harmonic* $2p$ *forms* on M. We need the following theorem.

Theorem 0.3.1. *(Hodge) Under the previous assumptions, the map $w \mapsto [w]$ defines an isomorphism of vector spaces*

$$\mathcal{H}^p(M) \to H^p_{dR}(M)$$

Remark 1. The space $\mathcal{H}^p(M)$ depends on the metric we choose. However, the dimension of this space is a topological invariant and independent of the choice of the metric.

Thus we get

$$\dim \ker(d + d^*)|_{\Omega^{ev}} = \sum_p \dim H^{2p}_{dR}(M).$$

Similarly we have:

$$\dim \ker(d + d^*)|_{\Omega^{odd}} = \sum_p \dim H^{2p+1}_{dR}(M).$$

This shows that

$$index(d + d^*) = \sum_p (-1)^p \dim H^p_{dR}(M) = \chi(M).$$

Now, we can consider the following question: is there a local formula for $\chi(M)$? More precisely we want to know if there is a naturally defined cohomology class whose integral over M gives the Euler characteristic of M. An affirmative answer to this question is given by the celebrated s Gauss-Bonnet-Chern theorem:

Theorem 0.3.2. *Let M be a closed oriented Riemannian manifold of dimension $2n$. Its Euler characteristic can be expressed as*

$$\chi(M) = (2\pi)^{-2n} \int_M Pf(\Omega).$$

Here, working in a coordinate system, $\Omega = (w_{ij})$ is an skew symmetric matrix of two forms (the curvature matrix of M), and the Pfaffian Pf is an invariant polynomial on the space of skew symmetric $2n \times 2n$ matrices. The Pfaffian is an square root of the determinant in the sense that $Pf(\Omega)^2 = det(\Omega)$. The first few are

$$\mathrm{pf}\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a.$$

$$\mathrm{pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + dc.$$

$$\mathrm{pf}\begin{bmatrix} 0 & a_1 & 0 & 0 & & & \\ -a_1 & 0 & b_1 & 0 & & & \\ 0 & -b_1 & 0 & a_2 & & & \\ 0 & 0 & -a_2 & \ddots & \ddots & & \\ & & & \ddots & \ddots & b_{n-1} & \\ & & & & -b_{n-1} & 0 & a_n \\ & & & & & -a_n & 0 \end{bmatrix} = a_1 a_2 \cdots a_n.$$

In general, if $A = (a_{ij})$ is an skew symmetric $2n \times 2n$ matrix, its Pfaffian is defined by

$$\mathrm{pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}$$

The Gauss-Bonnet-Chern theorem is a prototype of an index theorem. An invariant of a manifold defined as the index of an elliptic operator, the *analytic index*, is expressed in terms of the integral of a differential form, called a *topological index*. Thus the above result can be expressed as

$$\text{analytic index} \;=\; \text{topological index}$$

Remark 2. In the previous theorem if $dim(M) = 2$, we get the original Gauss-Bonnet theorem which relates the Euler characteristic of a surface to its total Gaussian curvature:

$$\chi(M) = \frac{1}{2\pi} \int_M K \, \mathrm{d}vol_g.$$

Here g is the Riemannian metric on M, and K is the Gauss curvature (which is half of the scalar curvature).

0.4 The signature theorem

A similar approach works for *Hirzebruch's signature theorem.* Let M be a closed oriented manifold of dimension $4n$. Consider the *intersection pairing* on middle forms $\Omega^{2n}(M)$:

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta.$$

It induces a symmetric bilinear form on middle cohomology $H^{2n}(M)$. The signature of M, $\sigma(M)$, is defined as the signature of this bilinear form.

Recall the definition of the *signature* of a bilinear form B on a finite dimensional vector space V. Let b_+ be the dimension of the largest subspace on which B is positive definite and b_- the dimension of the largest subspace on which B is negative definite. The signature of B is defined as the number $b_+ - b_-$.

Using Hodge theory, one shows that $\sigma(M)$ is also an index of an operator. Let $\Omega^p_{\mathbb{C}}(M)$ denote the complexified p-forms on M. Define an operator $\gamma : \Omega^p_{\mathbb{C}}(M) \to \Omega^{n-p}_{\mathbb{C}}(M)$ by

$$\gamma(\omega) = i^{p(p-1)+n} \star \omega, \qquad \omega \in \Omega^p(M)$$

Let $\Omega_+(M)$ (resp. $\Omega_-(M)$) denote the $+1$ (resp. -1) eigenspace of γ. Then using Hodge theory again one can show that the index of elliptic operator

$$d + d^* : \Omega_+(M) \to \Omega_-(M)$$

is equal to the signature of M:

$$index(d + d^*) = \sigma(M)$$

The Hirzebruch signature theorem exprsesses this analytic index as a topological index by the formula

$$\sigma(M) = \int_M L_n(p_1, \ldots, p_n)$$

where p_i's are Pontryagin classes and L is the Hirzebruch L-polynomial. The first are

$$\begin{aligned} L_0 &= 1 \\ L_1 &= \tfrac{1}{3} p_1 \\ L_2 &= \tfrac{1}{45}(7p_2 - p_1^2) \\ L_3 &= \tfrac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3) \end{aligned}$$

0.5 Riemann-Roch-Hirzebruch theorem

Let M be a compact complex manifold and E a holomorphic vector bundle on M. Let $\Omega^{(0,p)}(M,E)$ denote the space of $(0,p)-$ forms on M with coefficients in E. The holomorphic structure on E defines a differential operator

$$\bar{\partial}_E : \Omega^{(0,p)}(M,E) \to \Omega^{(0,p+1)}(M,E).$$

Like before we can assemble the even and odd parts of this Dolbeaux complex to get and elliptic PDE

$$\bar{\partial}_E : \bigoplus_p \Omega^{(0,2p)}(M,E) \to \bigoplus_p \Omega^{(0,2p+1)}(M,E).$$

The holomorphic Euler characteristic of E is defined as the index of this operator

$$\chi(E) = index\, \bar{\partial}_E$$

Using Hodge theory one shows that this definition coincides with the definition based on sheaf cohomology.

The Riemann-Roch theorem of Hirzebrch gives a formula for this number in terms of topological invariants:

$$\chi(E) = \int_M \mathrm{ch}(E)\, \mathrm{td}(X).$$

0.6 An odd index theorem

The index theorems we have discussed so far are all on even dimensional manifolds. In fact the index of any elliptic differential operator on an odd dimensional manifold is necessarily zero. One way to see this would be to express the index as the super trace of the heat kernel and use the general form of the asymptotic expansion of the heat trace. To get examples on odd dimensional manifolds, one can look at pseudodifferential operators. Here is a the simplest example.

Let $H = L^2(S^1)$. Any continuous function $f : S^1 \to \mathbb{C}$ defines a Toeplitz operator

$$T_f = PfP : H^+ \to H^+,$$

where H^+ is the Hardy space of functions whose negative Fourier coefficients vanish and $P : H \to H^+$ is the corresponding projection. Then it can be shown that T_f is a Fredholm operator if and only if f is nowhere zero. The following standard result, known as the Gohberg–Krein index theorem, computes the index of the Toeplitz operator in terms of the winding number of f:

$$\mathrm{index}(PfP) = -W(f,0).$$

To prove this formula notice that both sides are homotopy invariant. For the left-hand side this is a consequence of the homotopy invariance of the Fredholm index while for the right-hand side it is a standard fact about the winding number. Also, both sides are additive. Therefore it suffices to show that the two sides coincide on the generator of $\pi_1(S^1)$, i.e., for $f(z) = z$. Then PzP is easily seen to be the forward shift operator given by $PzP(e_n) = e_{n+1}$ in the Fourier basis. Clearly then $\text{index}(PzP) = -1 = -W(z, 0)$.

When f is smooth we have the following well-known formula for the winding number:

$$W(f, 0) = \frac{1}{2\pi i}\int f^{-1}\, df = \frac{1}{2\pi i}\varphi(f^{-1}, f),$$

where φ is the cyclic 1-cocycle on $C^\infty(S^1)$ defined by $\varphi(f, g) = \int f\, dg$. Since this cyclic cocycle is the Connes–Chern character of the Fredholm module (H, F), the above equation can be written as

$$\langle [(H,\, F)],\, [f]\rangle = \frac{1}{2\pi i}\langle \text{Ch}^{\text{odd}}(H, F),\, \text{Ch}_{\text{odd}}(f)\rangle,$$

where the pairing on the right-hand side is between cyclic cohomology and homology. As we shall prove next, this is a special case of a very general index formula of Connes.

0.7 Connes' noncommutative index theorem

We saw that the gist of an index theorem is the equality of an analytic index, defined as the index of a Fredholm operator, with a topological index, defined as the integration of a cohomology class over a homology cycle. To formulate an index theorem on a noncommutative space one must reformulative the analytic and topological and index in a new fashion.

Connes' index formulae that we hsall prove in these lectures (Theorem 0.20.1 and Proposition 0.20.1) acheives this goal as the following commutative diagram. In fact many aspects of noncommutative geometry are beautifully displayed in this diagram.

$$\begin{array}{ccccc} \mathfrak{K}^*(A) & \times & K_*(A) & \xrightarrow{\text{index}} & \mathbb{Z} \\ \Big\downarrow{\scriptstyle \text{Ch}^*} & & \Big\downarrow{\scriptstyle \text{Ch}_*} & & \Big\downarrow \\ HP^*(A) & \times & HP_*(A) & \longrightarrow & \mathbb{C}. \end{array} \qquad (1)$$

In this diagram:

1) A is an algebra which may very well be noncommutative.

2) $\mathfrak{K}^*(A)$ is the *set* of even resp. odd finitely summable Fredholm modules over A. It is closely related to the K-homology of A.

3) $K_*(A)$ is the algebraic K-theory of A.

4) Ch^* is Connes–Chern character in K-homology.

5) Ch_* is Connes–Chern character in K-theory.

6) $HP^*(A)$ is the periodic cyclic cohomology of A and $HP_*(A)$ is the periodic cyclic homology of A.

7) The top row is the *analytic index map*. It computes the Fredholm index of a Fredholm module twisted by a K-theory class.

8) The bottom row is the natural pairing between cyclic cohomology and homology. Once composed with vertical arrows it gives the *topological index maps*

$$[(H, F, \gamma)] \times [e] \mapsto \langle \mathrm{Ch}^0(H, F, \gamma), \mathrm{Ch}_0(e) \rangle,$$

$$[(H, F)] \times [u] \mapsto \langle \mathrm{Ch}^1(H, F), \mathrm{Ch}_1(u) \rangle.$$

So the commutativity of the diagram amounts to the following equality:

$$\boxed{\text{Topological Index} = \text{Analytic Index}} \tag{2}$$

Notice that the Atiyah–Singer index theorem amounts to an equality of the above type, where in this classical case $A = C^\infty(M)$ is a commutative algebra.

We can summarize early the earliest development of NCG as a way of extending (2) beyond its classical realm of manifolds and differential operators on them, to a noncommutative world. The following ingredients were needed:

1. What is a noncommutative space and how to construct one? A major source of noncommutative spaces is noncommutative quotients, replacing bad quotients by groupoid algebras.

2. Cohomological apparatus. This includes noncommutative analogues of topological invariants such as K-theory, K-homology, de Rham cohomology, and Chern character maps. Cyclic cohomology and the theory of characteristic classes in noncommutative geometry as we discuss later in these lectures.

The diagram (1) should be seen as the prototype of a series of results in noncommutative geometry that aims at expressing the analytic index by a topological formula. In the next step it would be desirable to have a *local* expression for the topological index, that is, for the Connes–Chern character Ch^i. The *local index formula* of Connes and Moscovici [25] solves this problem by replacing the characteristic classes $\mathrm{Ch}^i(H, F)$ by a cohomologous cyclic cocycle $\mathrm{Ch}^i(H, D)$. Here D is an unbounded operator that defines a refinement of the notion of Fredholm module to that of a *spectral triple*, and F is the *phase* of D. We won't discuss this aspect of noncommutative index theory in these lectures.

Lecture 2: Hochschild cohomology

Roughly speaking, Hochschild homology is the noncommutative analogue of differential forms on a manifold. This is made precise by Hochschild-Kostant-Rosenberg-Connes theorem. This theory was introduced by Hochschild to classify square zero extensions of an associative algebra. Group homology and Lie algebra homology are examples of Hochschild homology.

0.8 Hochschild cohomology

Let A be a unital algebra over $\mathbb{C}$ and M be an A-bimodule. Thus M is a left and a right A-module and the two actions are compatible in the sense that $a(mb) = (am)b$ for all a, b in A and m in M. The *Hochschild cochain complex of A with coefficients in M*,

$$\boxed{C^0(A, M) \xrightarrow{\delta} C^1(A, M) \xrightarrow{\delta} C^2(A, M) \xrightarrow{\delta} \cdots} \tag{3}$$

denoted $(C^*(A, M), \delta)$, is defined by

$$C^0(A, M) = M, \quad C^n(A, M) = \mathrm{Hom}(A^{\otimes n}, M), \quad n \geq 1,$$

where the differential $\delta\colon C^n(A, M) \to C^{n+1}(A, M)$ is given by

$$\begin{aligned}
(\delta m)(a) &= ma - am, \\
(\delta f)(a_1, \ldots, a_{n+1}) &= a_1 f(a_2, \ldots, a_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^{i+1} f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\
&\quad + (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}.
\end{aligned}$$

Here $m \in M = C^0(A, M)$, and $f \in C^n(A, M)$, $n \geq 1$.

One checks that

$$\delta^2 = 0.$$

The cohomology of the complex $(C^*(A, M), \delta)$ is by definition the *Hochschild cohomology* of the algebra A with coefficients in the A-bimodule M and will be denoted by $H^n(A, M)$, $n = 0, 1, 2, \ldots$.

Among all bimodules over an algebra A, the following two bimodules play an important role.

1) $M = A$, with bimodule structure $a(b)c = abc$ for all a, b, c in A. In this case the Hochschild complex $C^*(A, A)$ is also known as the *deformation* or *Gerstenhaber complex* of A. It plays an important role in deformation theory of associative algebras pioneered by Gerstenhaber [37]. For example, it can be shown that $H^1(A, A)$ is the space of *derivations* from $A \to A$ modulo inner derivations, $H^2(A, A)$ is the space of equivalence classes of *infinitesimal deformations* of A and $H^3(A, A)$ is the *space of obstructions* for deformations of A (cf. Section 3.3).

2) $M = A^* := \mathrm{Hom}(A, \mathbb{C})$, the linear dual of A, with A-bimodule structure defined by

$$(afb)(c) = f(bca)$$

for all a, b, c in A and f in A^*. This bimodule is relevant to cyclic cohomology. Indeed, as we shall see later in this chapter, the Hochschild groups $H^n(A, A^*)$ and the cyclic cohomology groups $HC^n(A)$ enter into a long exact sequence. Using the identification

$$\mathrm{Hom}(A^{\otimes n}, A^*) \simeq \mathrm{Hom}(A^{\otimes(n+1)}, \mathbb{C}), \quad f \mapsto \varphi,$$

$$\varphi(a_0, a_1, \ldots, a_n) = f(a_1, \ldots, a_n)(a_0),$$

the Hochschild differential δ is transformed into a differential, denoted b, given by

$$\begin{aligned}(b\varphi)(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1})\\ + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \ldots, a_n).\end{aligned}$$

Thus for $n = 0, 1, 2$ we have the following formulas for b:

$$\begin{aligned}(b\varphi)(a_0, a_1) &= \varphi(a_0 a_1) - \varphi(a_1 a_0),\\ (b\varphi)(a_0, a_1, a_2) &= \varphi(a_0 a_1, a_2) - \varphi(a_0, a_1 a_2) + \varphi(a_2 a_0, a_1),\\ (b\varphi)(a_0, a_1, a_2, a_3) &= \varphi(a_0 a_1, a_2, a_3) - \varphi(a_0, a_1 a_2, a_3)\\ &\quad + \varphi(a_0, a_1, a_2 a_3) - \varphi(a_3 a_0, a_1, a_2).\end{aligned}$$

From now on the Hochschild complex $C^*(A, A^*)$ will be simply denoted by $C^*(A)$ and the Hochschild cohomology $H^*(A, A^*)$ by $HH^*(A)$.

Example 0.8.1. We give a few examples of Hochschild cohomology, starting in low dimensions.

1. $n = 0$. It is clear that

$$H^0(A, M) = \{m \in M;\ ma = am \text{ for all } a \in A\}.$$

In particular for $M = A^*$,

$$H^0(A,\, A^*) = \{f\colon A \to \mathbb{C};\ f(ab) = f(ba) \text{ for all } a, b \in A\}$$

is the *space of traces* on A.

2. $n = 1$. A Hochschild 1-*cocycle* $f \in C^1(A,\, M)$ is simply a *derivation*, i.e., a $\mathbb{C}$-linear map $f\colon A \to M$ such that

$$f(ab) = af(b) + f(a)b$$

for all a, b in A. A 1-cocycle is a *coboundary* if and only if the corresponding derivation is *inner*, that is there should exists an m in M such that $f(a) = ma - am$ for all a in A. Therefore

$$H^1(A,\, M) = \frac{\text{derivations}}{\text{inner derivations}}.$$

Sometimes this is called the space of *outer derivations* of A with values in the A-bimodule M. In view of Exercise 0.8.6, for an algebra A, commutative or not, we can think of $\mathrm{Der}(A,\, A)$ as the Lie algebra of noncommutative vector fields on the noncommutative space represented by A. Notice that, unless A is commutative, $\mathrm{Der}(A,\, A)$ need not be an A-module.

3. $n = 2$. One can show that $H^2(A, M)$ classifies *abelian extensions* of A by M. Let A be a unital algebra and M be an A-bimodule. By definition, an abelian extension of A by M is an exact sequence of algebras

$$0 \to M \to B \to A \to 0$$

such that B is unital, M has trivial multiplication (i.e., $M^2 = 0$), and the induced A-bimodule structure on M coincides with the original bimodule structure. Two such extensions (M, B, A) and (M, B', A) are called isomorphic if there is a unital algebra map $f\colon B \to B'$ which induces identity maps on M and A. (Notice that if such an f exists then it is necessarily an isomorphism.) Let $E(A, M)$ denote the set of isomorphism classes of such extensions. We define a natural bijection

$$E(A, M) \simeq H^2(A, M)$$

as follows. Given an extension as above, let $s\colon A \to B$ be a linear splitting for the projection $B \to A$, and let $f\colon A \otimes A \to M$ be its *curvature*, defined by

$$f(a, b) = s(ab) - s(a)s(b)$$

for all a, b in A. One can easily check that f is a Hochschild 2-cocycle and its class is independent of the choice of the splitting s. In the other direction, given a 2-cochain $f\colon A \otimes A \to M$, we try to define a multiplication on $B = A \oplus M$ via

$$(a, m)(a', m') = (aa', am' + ma' + f(a, a')).$$

It can be checked that this defines an associative multiplication if and only if f is a 2-cocycle. The extension associated to a 2-cocycle f is the extension

$$0 \to M \to A \oplus M \to A \to 0.$$

It can be checked that these two maps are bijective and inverse to each other.

4. A simple computation shows that when $A = \mathbb{C}$ is the ground field we have

$$HH^0(\mathbb{C}) = \mathbb{C} \quad \text{and} \quad HH^n(\mathbb{C}) = 0 \quad \text{for } n \geq 1.$$

Example 0.8.2. Let M be a closed (i.e., compact without boundary), smooth, oriented, n-dimensional manifold and let $A = C^\infty(M)$ denote the algebra of complex valued smooth functions on M. For $f^0, \ldots, f^n \in C^\infty(M)$, let

$$\varphi(f^0, \ldots, f^n) = \int_M f^0\, df^1 \ldots df^n.$$

The $(n+1)$-linear cochain $\varphi\colon A^{\otimes(n+1)} \to \mathbb{C}$ has three properties: it is *continuous* with respect to the natural Fréchet space topology of A (cf. Section 0.11 for more on this point); it is a *Hochschild cocycle*; and it is a *cyclic cochain* (cf. Section 0.12 for more on this). The Hochschild cocycle property that concerns us here, $b\varphi = 0$, can be checked as follows:

$$\begin{aligned}(b\varphi)(f^0, \ldots, f^{n+1}) &:= \sum_{i=0}^{n} (-1)^i \varphi(f^0, \ldots, f^i f^{i+1}, \ldots, f^{n+1}) \\ &\quad + (-1)^{n+1}\varphi(f^{n+1} f^0, \ldots, \ldots, f^n) \\ &= \sum_{i=0}^{n} (-1)^i \int_M f^0\, df^1 \ldots d(f^i f^{i+1}) \ldots df^{n+1} \\ &\quad + (-1)^{n+1} \int_M f^{n+1} f^0\, df^1 \ldots df^n \\ &= 0\end{aligned}$$

for all $f^0, \ldots, f^{n+1} \in A$. Here we used the Leibniz rule for the de Rham differential d and the graded commutativity of the algebra $(\Omega^* M, d)$ of differential forms on M.

We have thus associated a Hochschild cocycle to the orientation cycle of the manifold. This construction admits a vast generalization, as we explain now. Let

$$\Omega_p M := \mathrm{Hom}_{\mathrm{cont}}(\Omega^p M, \mathbb{C}) \tag{4}$$

denote the *continuous linear dual* of the space of p-forms on M. Here, the (locally convex) topology of $\Omega^p M$ is defined by the sequence of seminorms

$$\|\omega\|_n = \sup |\partial^\alpha\, \omega_{i_1, \ldots, i_p}|, \quad |\alpha| \leq n,$$

where the supremum is over a fixed, finite, coordinate cover for M, and over all partial derivatives ∂^α of total degree at most n of all components $\omega_{i_1,\ldots,i_p}$ of ω. Elements of $\Omega_p M$ are called *de Rham p-currents* on M. For $p=0$ we recover the notion of a *distribution* on M. Since the de Rham differential $d\colon \Omega^k M \to \Omega^{k+1} M$, $k=0,1,\ldots$, is continuous in the topology of differential forms, by dualizing it we obtain differentials $d^*\colon \Omega_k M \to \Omega_{k-1} M$, $k=1,2,\ldots$ and the *de Rham complex of currents* on M:

$$\Omega_0 M \xleftarrow{d^*} \Omega_1 M \xleftarrow{d^*} \Omega_2 M \xleftarrow{d^*} \cdots .$$

The homology of this complex is called the *de Rham homology* of M and we shall denote it by $H_n^{\mathrm{dR}}(M)$, $n=0,1,\ldots$.

It is easy to check that for *any* m-current C, closed or not, the cochain φ_C defined by

$$\varphi_C(f^0, f^1, \ldots, f^m) := \langle C,\, f^0 df^1 \ldots df^m \rangle$$

is a Hochschild cocycle on A. As we shall explain in Section 3.4, φ_C is continuous in the natural topology of $A^{\otimes(m+1)}$ and we obtain a canonical map

$$\boxed{\Omega_m M \to HH^m_{\mathrm{cont}}(C^\infty(M))}$$

from the space of m-currents on M to the continuous Hochschild cohomology of $C^\infty(M)$. By a theorem of Connes [19] this map is an isomorphism. We refer to Section 3.5 for more details and a dual statement relating differential forms with Hochschild *homology*. The corresponding statement for the algebra of regular functions on a smooth affine variety, the Hochschild–Kostant–Rosenberg theorem, will be discussed in that section as well.

Exercise 0.8.1. Let $A_1 = \mathbb{C}[x, \frac{d}{dx}]$ denote the *Weyl algebra* of differential operators with polynomial coefficients, where the product is defined as the composition of operators. Equivalently, A_1 is the unital universal algebra generated by elements x and $\frac{d}{dx}$ with relation $\frac{d}{dx}x - x\frac{d}{dx} = 1$. Show that $HH^0(A_1) = 0$; that is, A_1 carries no nonzero trace.

Exercise 0.8.2. Show that any derivation of the Weyl algebra $A_1 = \mathbb{C}\,[x, \frac{d}{dx}]$ is inner, i.e., $H^1(A_1, A_1) = 0$.

Exercise 0.8.3. Show that any derivation of the algebra $C(X)$ of continuous functions on a compact Hausdorff space X is zero. (Hint: If $f = g^2$ and $g(x) = 0$ for some $x \in X$ then, for any derivation δ, $(\delta f)(x) = 0$.)

Exercise 0.8.4. Show that any derivation of the matrix algebra $M_n(\mathbb{C})$ is inner. (This was proved by Dirac in his first paper on quantum mechanics [34], where derivations are called *quantum differentials*).

Exercise 0.8.5. Let $Z(A)$ denote the center of the algebra A. Show that the Hochschild groups $H^n(A, M)$ are $Z(A)$-modules.

Exercise 0.8.6 (Derivations and vector fields)**.** Let $U \subset \mathbb{R}^n$ be an open set and let

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$$

be a smooth vector field on U. Define a derivation $\delta_X \colon C^\infty(U) \to C^\infty(U)$ by

$$\delta_X(f) = \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i}.$$

Show that the map $X \mapsto \delta_X$ defines a 1-1 correspondence between vector fields on U and derivations of $C^\infty(U)$ to itself. Under this isomorphism, the bracket of vector fields corresponds to commutators of derivations:

$$\delta_{[X,\,Y]} = [\delta_X,\, \delta_Y].$$

Fix a point $m \in U$ and define an A-module structure on $\mathbb{C}$ by the map $f \otimes 1 \mapsto f(m)$. Show that the set $\mathrm{Der}(C^\infty(U), \mathbb{C})$ of $\mathbb{C}$-valued derivations of $C^\infty(U)$ is canonically isomorphic to the (complexified) tangent space of U at m. Extend these correspondences to arbitrary smooth manifolds. (These considerations form the beginnings of a purely algebraic approach to some *'soft'* aspects of differential geometry including differential forms and tensor analysis, connection and curvature formalism and Chern–Weil theory and is part of 'differential geometry over commutative algebras'. It can also be adapted to algebraic geometry.)

0.9 Hochschild cohomology as a derived functor

The original complex (3) that we used to define the Hochschild cohomology is rarely useful for computations. Instead, the fact that Hochschild cohomology is a *derived functor* will allows us, in specific cases, to replace the standard complex (3) by a much smaller complex and to compute the Hochschild cohomology. In this section we show that Hochschild cohomology is a derived functor; more precisely it is an Ext functor. References for the general theory of derived functors and homological algebra include [14], and [56].

Let A^{op} denote the *opposite algebra* of an algebra A. Thus, as a vector space $A^{\mathrm{op}} = A$ and the new multiplication is defined by $a \cdot b := ba$. There is a one-to-one correspondence between A-bimodules and left $A \otimes A^{\mathrm{op}}$-modules defined by

$$(a \otimes b^{\mathrm{op}})m = amb.$$

Define a functor from the category of left $A \otimes A^{\mathrm{op}}$-modules to the category of complex vector spaces by

$$M \mapsto \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A,\, M) = \{m \in M;\ ma = am \text{ for all } a \in A\} = H^0(A,\, M).$$

We show that Hochschild cohomology is the left derived functor of the functor $M \rightsquigarrow H^0(A, M)$. We assume that A is *unital*. Since A is naturally a left $A \otimes A^{\mathrm{op}}$-module, we can consider its *bar resolution*. It is defined by

$$0 \leftarrow A \xleftarrow{b'} B_1(A) \xleftarrow{b'} B_2(A) \xleftarrow{b'} \cdots, \tag{5}$$

where $B_n(A) = A \otimes A^{\mathrm{op}} \otimes A^{\otimes n}$ is the free left $A \otimes A^{\mathrm{op}}$-module generated by $A^{\otimes n}$. The differential b' is defined by

$$\begin{aligned} b'(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) &= aa_1 \otimes b \otimes a_2 \otimes \cdots \otimes a_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^i (a \otimes b \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad + (-1)^n (a \otimes a_n b \otimes a_1 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

Define the operators $s\colon B_n(A) \to B_{n+1}(A)$, $n \geq 0$, by

$$s(a \otimes b \otimes a_1 \otimes \cdots \otimes a_n) = 1 \otimes b \otimes a \otimes a_1 \otimes \cdots \otimes a_n.$$

One checks that

$$b's + sb' = \mathrm{id},$$

which shows that $(B(A), b')$ is acyclic and hence is a free resolution of A as a left $A \otimes A^{\mathrm{op}}$-module. Now, for any A-bimodule M we have an isomorphism of cochain complexes

$$\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(B(A), M) \simeq (C^*(A, M), \delta),$$

which shows that Hochschild cohomology is the left derived functor of the Hom functor:

$$H^n(A, M) \simeq \mathrm{Ext}^n_{A \otimes A^{\mathrm{op}}}(A, M) \quad \text{for all } n \geq 0.$$

One can therefore use any projective resolution of A, or any injective resolution of M, as a left $A \otimes A^{\mathrm{op}}$-module to compute the Hochschild cohomology groups.

Before proceeding further let us recall the definition of the *Hochschild homology* of an algebra A with coefficients in a bimodule M. The *Hochschild homology complex of A with coefficients in M* is the complex

$$C_0(A, M) \xleftarrow{\delta} C_1(A, M) \xleftarrow{\delta} C_2(A, M) \xleftarrow{\delta} \cdots \tag{6}$$

denoted by $(C_*(A, M), \delta)$, where

$$C_0(A, M) = M \quad \text{and} \quad C_n(A, M) = M \otimes A^{\otimes n}, \qquad n = 1, 2, \ldots,$$

and the *Hochschild boundary* $\delta\colon C_n(A, M) \to C_{n-1}(A, M)$ is defined by

$$\begin{aligned} \delta(m \otimes a_1 \otimes \cdots \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &\quad + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_n. \end{aligned}$$

The Hochschild homology of A with coefficients in M is, by definition, the homology of the complex $(C_*(A,\, M),\, \delta)$. We denote this homology by $H_n(A,\, M)$, $n = 0, 1, \ldots$. It is clear that

$$H_0(A,\, M) = M/[A,\, M],$$

where $[A,\, M]$ is the $\mathbb{C}$-linear subspace of M spanned by commutators $am - ma$ for a in A and m in M.

The following facts are easily established:

1) Hochschild homology, $H_*(A,\, M)$, is the right derived functor of the functor $M \rightsquigarrow A \otimes_{A\otimes A^{\mathrm{op}}} M = H_0(A, M)$ from the category of left $A \otimes A^{\mathrm{op}}$-modules to the category of complex vector spaces, i.e.,

$$H_n(A,\, M) \simeq \mathrm{Tor}_n^{A\otimes A^{\mathrm{op}}}(A, M).$$

For the proof one can simply use the bar resolution (5) as we did for cohomology.

2) (Duality) Let $M^* = \mathrm{Hom}(M, \mathbb{C})$. It is an A-bimodule via $(afb)(m) = f(bma)$. One checks that the natural isomorphism

$$\mathrm{Hom}(A^{\otimes n},\, M^*) \simeq \mathrm{Hom}(M \otimes A^{\otimes n}, \mathbb{C}), \quad n = 0, 1, \ldots$$

is compatible with differentials. Thus, since we are over a field of characteristic 0, we have natural isomorphisms

$$H^n(A,\, M^*) \simeq (H_n(A,\, M))^*, \quad n = 0, 1, \ldots$$

From now on the Hochschild homology groups $H_*(A, A)$ will be denoted by $HH_*(A)$. In view of the above duality, we have the isomorphisms

$$HH^n(A) \simeq HH_n(A)^*, \quad n \geq 0,$$

where by our earlier convention $HH^n(A)$ stands for $H^n(A, A^*)$.

Example 0.9.1. Let $A = \mathbb{C}[x]$ be the algebra of polynomials in one variable. It is easy to check that the following complex is a resolution of A as a left $A \otimes A$-module:

$$0 \leftarrow \mathbb{C}[x] \xleftarrow{\varepsilon} \mathbb{C}[x] \otimes \mathbb{C}[x] \xleftarrow{d} \mathbb{C}[x] \otimes \mathbb{C}[x] \otimes \mathbb{C} \leftarrow 0, \tag{7}$$

where the differentials are the unique $A \otimes A$-linear extensions of the maps

$$\varepsilon(1 \otimes 1) = 1, \quad d(1 \otimes 1 \otimes 1) = x \otimes 1 - 1 \otimes x. \tag{8}$$

To check its acyclicity, notice that it is isomorphic to the complex

$$0 \leftarrow \mathbb{C}[x] \xleftarrow{\varepsilon} \mathbb{C}[x, y] \xleftarrow{d} \mathbb{C}[x, y] \leftarrow 0,$$

where now

$$\varepsilon(f(x, y)) = f(x, x), \quad d(f(x, y)) = (x - y)f(x, y).$$

By tensoring this resolution with the right $A \otimes A$-module A, we obtain a complex with zero differentials

$$0 \leftarrow \mathbb{C}[x] \xleftarrow{0} \mathbb{C}[x] \leftarrow 0$$

and hence

$$HH_i(\mathbb{C}[x]) \simeq \begin{cases} \mathbb{C}[x] & \text{if } i = 0, 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

The complex (7) is a simple example of a *Koszul resolution.* In the next example we generalize it to polynomials in several variables.

Example 0.9.2. Let $A = \mathbb{C}[x_1, \ldots, x_n]$ be the algebra of polynomials in n variables. Let V be an n-dimensional complex vector space over. The Koszul resolution of A, as a left $A \otimes A$-module, is defined by

$$0 \leftarrow A \xleftarrow{\varepsilon} A \otimes A \xleftarrow{d} A \otimes A \otimes \Omega^1 \leftarrow \cdots \leftarrow A \otimes A \otimes \Omega^i \leftarrow \cdots \leftarrow A \otimes A \otimes \Omega^n \leftarrow 0, \quad (9)$$

where $\Omega^i = \bigwedge^i V$ is the i-th exterior power of V. The differentials ε and d are defined in (8). d has a unique extension to a graded derivation of degree -1 on the graded commutative algebra $A \otimes A \otimes \bigwedge V$. Notice that $A \simeq S(V)$, the symmetric algebra of the vector space V.

Let $K(S(V))$ denote the Koszul resolution (9). To show that it is exact we notice that

$$K(S(V \oplus W)) \simeq K(S(V) \otimes K(S(W))).$$

Since the tensor product of two exact complexes is again exact (notice that we are over a field of characteristic zero), the exactness of $K(S(V))$ can be reduced to the case where V is 1-dimensional, which was treated in the last example. See Exercise 0.9.6 for an explicit description of the resolution (9).

As in the one dimensional case, the differentials in the complex $A \otimes_{A \otimes A} K(S(V))$ are all zero and we obtain

$$\begin{aligned} HH_i(S(V)) &= \operatorname{Tor}_i^{S(V) \otimes S(V)}(S(V), S(V) \\ &= S(V) \otimes \textstyle\bigwedge^i V. \end{aligned}$$

The right-hand side is isomorphic to the module of algebraic differential forms on $S(V)$. So we can write this result as

$$HH_i(S(V)) = \Omega^i(S(V)),$$

which is a special case of the Hochschild–Kostant–Rosenberg theorem mentioned before. More generally, if M is a *symmetric* A-bimodule, the differentials of $M \otimes_{A \otimes A} K(S(V))$ vanish and we obtain

$$H_i(S(V), M) \simeq M \otimes \textstyle\bigwedge^i V, \quad i = 0, 1, \ldots, n,$$

and 0 otherwise.

Example 0.9.3. In Section 0.11 we shall define the continuous analogues of Hochschild and cyclic (co)homology as well as Tor and Ext functors. Here is a simple example. The continuous analogue of the resolution (7) for the topological algebra $A = C^\infty(S^1)$ is the *topological Koszul resolution*

$$0 \leftarrow C^\infty(S^1) \xleftarrow{\varepsilon} C^\infty(S^1) \hat{\otimes} C^\infty(S^1) \xleftarrow{d} C^\infty(S^1) \hat{\otimes} C^\infty(S^1) \otimes \mathbb{C} \leftarrow 0, \qquad (10)$$

with differentials given by (8). Here $\hat{\otimes}$ denotes the *projective* tensor product of locally convex spaces (cf. Section 0.11 for definitions). To verify the exactness, the only non-trivial step is to check that $\ker \varepsilon \subset \operatorname{im} d$. To this end, notice that if we identify

$$C^\infty(S^1) \hat{\otimes} C^\infty(S^1) \simeq C^\infty(S^1 \times S^1),$$

the differentials are given by

$$(\varepsilon f)(x) = f(x, x), \quad (d_1 f)(x, y) = (x - y) f(x, y).$$

Now the homotopy formula

$$f(x, y) = f(x, x) - (x - y) \int_0^1 \frac{\partial}{\partial y} f(x,\, y + t(x - y))\, dt$$

shows that $\ker \varepsilon \subset \operatorname{im} d$. Alternatively, one can use Fourier series to establish the exactness (cf. Exercise 0.9.5).

To compute the continuous Tor functor, we apply the functor $- \hat{\otimes}_{A \hat{\otimes} A}$ to the above complex. We obtain

$$0 \leftarrow C^\infty(S^1) \xleftarrow{0} C^\infty(S^1) \leftarrow 0$$

and hence

$$HH_i^{\text{cont}}(C^\infty(S^1)) = \begin{cases} \Omega^i S^1 & \text{if } i = 0, 1, \\ 0 & \text{if } i \geq 2, \end{cases}$$

where $\Omega^i S^1 \simeq C^\infty(S^1) dx^i$ is the space of differential forms of degree i on S^1.

A similar computation, using a continuous version of Ext by applying the functor $\operatorname{Hom}^{\text{cont}}_{A \hat{\otimes} A}(-, A)$ gives

$$HH^i_{\text{cont}}(C^\infty(S^1)) = \begin{cases} \Omega_i S^1 & \text{if } i = 0, 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

Here $\Omega_i S^1 = (\Omega^i S^1)^*$, the continuous dual of i-forms, is the space of i-currents on S^1.

Notice how the identification $C^\infty(S^1) \hat{\otimes} C^\infty(S^1) \simeq C^\infty(S^1 \times S^1)$ played an important role in the above proof. The algebraic tensor product $C^\infty(S^1) \otimes C^\infty(S^1)$, on the other hand, is only *dense* in $C^\infty(S^1 \times S^1)$ and this makes it very difficult to write a resolution to compute the *algebraic* Hochschild groups of $C^\infty(S^1)$. In fact these groups are not known so far!

Example 0.9.4 (Cup product)**.** Let A and B be unital algebras. What is the relation between the Hochschild homology groups of $A \otimes B$ and those of A and B? One can construct (cf. [14], [56] for details) chain maps

$$C_*(A \otimes B) \to C_*(A) \otimes C_*(B),$$
$$C_*(A) \otimes C_*(B) \to C_*(A \otimes B)$$

inducing inverse isomorphisms. We obtain

$$HH_n(A \otimes B) \simeq \bigoplus_{p+q=n} HH_p(A) \otimes HH_q(B) \quad \text{for all } n \geq 0.$$

Now, if A is commutative, the multiplication $m\colon A \otimes A \to A$ is an algebra map and, in combination with the above map, induces an associative and graded commutative product on $HH_*(A)$.

Exercise 0.9.1. Let A and B be unital algebras. Give a direct proof of the isomorphism

$$HH_0(A \otimes B) \simeq HH_0(A) \otimes HH_0(B).$$

Dually, show that there is a natural map

$$HH^0(A) \otimes HH^0(B) \to HH^0(A \otimes B),$$

but it need not be surjective in general.

Exercise 0.9.2. Let

$$A = T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus \cdots,$$

be the tensor algebra of a vector space V. Show that the complex

$$0 \leftarrow A \xleftarrow{\varepsilon} A \otimes A^{\mathrm{op}} \xleftarrow{d} A \otimes A^{\mathrm{op}} \otimes V \leftarrow 0,$$

with differentials induced by

$$\varepsilon(1 \otimes 1) = 1, \quad d(1 \otimes 1 \otimes v) = v \otimes 1 - 1 \otimes v, \quad v \in V,$$

is a free resolution of A as a left $A \otimes A^{\mathrm{op}}$-module. Conclude that A has Hochschild homological dimension 1 in the sense that $H_n(A, M) = 0$ for all A-bimodules M and all $n \geq 2$. Compute $H_0(A, M)$ and $H_1(A, M)$.

Exercise 0.9.3 (Normalization)**.** Let M be an A-bimodule. A cochain $f\colon A^{\otimes n} \to M$ is called *normalized* if $f(a_1, \ldots, a_n) = 0$ whenever $a_i = 1$ for some i. Show that normalized cochains $C^*_{\mathrm{norm}}(A, M)$ form a subcomplex of the Hochschild complex $C^*(A, M)$ and that the inclusion

$$C^*_{\mathrm{norm}}(A, M) \hookrightarrow C^*(A, M)$$

is a quasi-isomorphism. (Hint: Introduce a normalized version of the bar resolution.)

Exercise 0.9.4. Let $A = \mathbb{C}[x]/(x^2)$ denote the algebra of *dual numbers*. Use the normalized Hochschild complex to compute $HH_*(A)$.

Exercise 0.9.5. Use Fourier series to show that the sequence (10) is exact.

Exercise 0.9.6. Let V be an n-dimensional vector space. Show that the following complex is a free resolution of $S(V)$, the symmetric algebra of V, as a left $S(V) \otimes S(V)$-module

$$S(V) \xleftarrow{\varepsilon} S(V^2) \xleftarrow{i_X} S(V^2) \otimes E_1 \xleftarrow{i_X} S(V^2) \otimes E_2 \xleftarrow{i_X} \cdots \xleftarrow{i_X} S(V^2) \otimes E_n \leftarrow 0,$$

where $E_k = \bigwedge^k V$, and i_X is the interior multiplication (contraction) with respect to the vector field

$$X = \sum_{i=1}^{n} (x_i - y_i) \frac{\partial}{\partial y_i}$$

on $V^2 = V \times V$. (Hint: Use the Cartan homotopy formula $di_X + i_X d = L_X$ to find a contracting homotopy for i_X.)

Exercise 0.9.7 (A resolution for the algebraic noncommutative torus [19])**.** Let

$$A = \mathbb{C}\langle U_1, U_2 \rangle / (U_1 U_2 - \lambda U_2 U_1)$$

be the universal unital algebra generated by invertible elements U_1 and U_2 with relation $U_1 U_2 = \lambda U_2 U_1$. We assume that $\lambda \in \mathbb{C}$ is not a root of unity. Let $\Omega^i = \bigwedge^i V$, where V is a 2-dimensional vector space with basis e_1 and e_2. Consider the complex of left $A \otimes A^{\mathrm{op}}$-modules

$$0 \leftarrow A \xleftarrow{\varepsilon} A \otimes A^{\mathrm{op}} \xleftarrow{d_0} A \otimes A^{\mathrm{op}} \otimes \Omega^1 \xleftarrow{d_1} A \otimes A^{\mathrm{op}} \otimes \Omega^2 \leftarrow 0, \tag{11}$$

where ε is the multiplication map and the other differentials are defined by

$$d_0(1 \otimes 1 \otimes e_j) = 1 \otimes U_j - U_j \otimes 1, \quad j = 1, 2,$$

$$d_1(1 \otimes 1 \otimes e_1 \wedge e_2) = (U_2 \otimes 1 - \lambda \otimes U_2) \otimes e_1 - (\lambda U_1 \otimes 1 - 1 \otimes U_1) \otimes e_2.$$

Show that (11) is a resolution of A as an $A \otimes A^{\mathrm{op}}$-module and use it to compute $HH_*(A)$.

Exercise 0.9.8. The Weyl algebra A_1 is defined in Exercise 0.8.1. By giving a 'small' resolution of length two for A_1 as a left $A_1 \otimes A_1^{\mathrm{op}}$-module show that

$$HH_i(A_1) \simeq \begin{cases} \mathbb{C} & \text{if } i = 2, \\ 0 & \text{if } i \neq 2. \end{cases}$$

Show that $HH_2(A_1)$ is generated by the class of the 2-cycle

$$1 \otimes p \otimes q - 1 \otimes q \otimes p + 1 \otimes 1 \otimes 1,$$

where $q = x$ and $p = \frac{d}{dx}$. Extend this result to higher order Weyl algebras $A_n = A_1^{\otimes n}$ and show that

$$HH_i(A_n) \simeq \begin{cases} \mathbb{C} & \text{if } i = 2n, \\ 0 & \text{if } i \neq 2n. \end{cases}$$

Can you give an explicit formula for the generator of $HH_{2n}(A_n)$?

0.10 Deformation theory

Let A be a unital complex algebra. An increasing *filtration* on A is an increasing sequence of subspaces of A, $F^i(A) \subset F^{i+1}(A)$, $i = 0, 1, 2, \ldots$, with $1 \in F^0(A)$, $\bigcup_i F^i(A) = A$, and

$$F^i(A)F^j(A) \subset F^{i+j}(A) \quad \text{for all } i,\, j.$$

Let $F^{-1}(A) = 0$. The *associated graded algebra* of a filtered algebra is the graded algebra

$$\mathrm{Gr}(A) = \bigoplus_{i \geq 0} \tfrac{F^i(A)}{F^{i-1}(A)}.$$

Definition 0.10.1. An *almost commutative* algebra is a filtered algebra whose associated graded algebra $\mathrm{Gr}(A)$ is commutative.

Being almost commutative is equivalent to the commutator condition

$$[F^i(A),\, F^j(A)] \subset F^{i+j-1}(A) \tag{12}$$

for all i, j. As we shall see, Weyl algebras and, more generally, algebras of differential operators on a smooth manifold, and universal enveloping algebras are examples of almost commutative algebras.

Let A be an almost commutative algebra. The original Lie algebra bracket $[x, y] = xy - yx$ on A induces a Lie algebra bracket $\{\ \}$ on $\mathrm{Gr}(A)$ via the formula

$$\{x + F^i, y + F^j\} := [x, y] + F^{i+j-2}.$$

Notice that by the almost commutativity assumption (12), $[x, y]$ is in $F^{i+j-1}(A)$ and $\mathrm{Gr}(A)$, with its grading shifted by one, is indeed a *graded* Lie algebra. The induced Lie bracket on $\mathrm{Gr}(A)$ is compatible with its multiplication in the sense that for all $a \in \mathrm{Gr}(A)$, the map $b \mapsto \{a,\, b\}$ is a derivation. The algebra $\mathrm{Gr}(A)$ is called the *semiclassical limit* of the almost commutative algebra A. It is an example of a Poisson algebra as we recall later in this section.

Notice that as *vector spaces*, $\mathrm{Gr}(A)$ and A are linearly isomorphic, but their algebra structures are different as $\mathrm{Gr}(A)$ is always commutative but A need not be commutative. A linear isomorphism

$$q\colon\ \mathrm{Gr}(A) \to A$$

can be regarded as a *'naive quantization map'*. Of course, linear isomorphisms always exist but they are hardly interesting. One usually demands more. For example one wants q to be a Lie algebra map in the sense that

$$q\{a,\, b\} = [q(a),\, q(b)] \tag{13}$$

for all a, b in $\mathrm{Gr}(A)$. This is one form of *Dirac's quantization rule*, going back to Dirac's paper [34]. One normally thinks of A as the algebra of quantum observables of a system acting as operators on a Hilbert space, and of $\mathrm{Gr}(A)$ as the algebra of classical observables of functions on the phase space. *No-go theorems*, e.g. the celebrated Groenewold–Van Hove Theorem (cf. [40] for discussions and precise statements; see also Exercise 0.10.2), states that, under reasonable irreducibility conditions, this is almost never possible. The remedy is to have q defined only for a special class of elements of $\mathrm{Gr}(A)$, or satisfy (13) only in an *asymptotic sense* as Planck's constant h goes to zero. As we shall discuss later in this section, this can be done in different ways, for example in the context of formal deformation quantization [53].

The notion of a Poisson algebra captures the structure of semiclassical limits.

Definition 0.10.2. Let P be a commutative algebra. A *Poisson structure* on P is a Lie algebra bracket $(a, b) \mapsto \{a, b\}$ on A such that for any $a \in A$, the map $b \mapsto \{a, b\}\colon A \to A$ is a derivation of A. That is, for all b, c in A we have

$$\{a, bc\} = \{a, b\}c + b\{a, c\}.$$

In geometric examples (see below) the vector field defined by the derivation $b \mapsto \{a, b\}$ is called the *Hamiltonian vector field* of the *Hamiltonian function* a.

Definition 0.10.3. A *Poisson algebra* is a pair $(P, \{\ ,\ \})$ where P is a commutative algebra and $\{\ ,\ \}$ is a Poisson structure on P.

We saw that the semiclassical limit $P = \mathrm{Gr}(A)$ of any almost commutative algebra A is a Poisson algebra. Conversely, given a Poisson algebra P one may ask if it is the semiclassical limit of an almost commutative algebra. This is one form of the problem of quantization of Poisson algebras, the answer to which for general Poisson algebras is negative.

Example 0.10.1. A *Poisson manifold* is a manifold M whose algebra of smooth functions $A = C^\infty(M)$ is a Poisson algebra (we should also assume that the bracket $\{\ ,\ \}$ is continuous in the Fréchet topology of A, or, equivalently, is a bidifferential operator). It is not difficult to see that all Poisson structures on A are of the form

$$\{f,\, g\} := \langle df \wedge dg,\, \pi\rangle,$$

where $\pi \in C^\infty(\bigwedge^2(TM))$ is a smooth 2-vector field on M. This bracket clearly satisfies the Leibniz rule in each variable and one checks that it satisfies the Jacobi identity if and only if $[\pi, \pi] = 0$, where the *Schouten bracket* $[\pi, \pi] \in C^\infty(\bigwedge^3(TM))$ is defined in local coordinates by

$$[\pi, \pi]_{ijk} = \sum_{l=1}^{n} \left(\pi_{lj} \frac{\partial \pi_{ik}}{\partial x_l} + \pi_{li} \frac{\partial \pi_{kj}}{\partial x_l} + \pi_{lk} \frac{\partial \pi_{ji}}{\partial x_l} \right).$$

The Poisson bracket in local coordinates is given by

$$\{f, g\} = \sum_{ij} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Symplectic manifolds are the simplest examples of Poisson manifolds. They correspond to non-degenerate Poisson structures. Recall that a *symplectic form* on a manifold is a non-degenerate closed 2-form on the manifold. Given a symplectic form ω, the associated Poisson bracket is given by

$$\{f, g\} = \omega(X_f, X_g),$$

where the vector field X_f is the *symplectic dual* of *df* and is defined by requiring that the equation $df(Y) = \omega(X_f, Y)$ holds for all smooth vector fields Y on M.

Let $C^\infty_{\text{poly}}(T^*M)$ be the algebra of smooth functions on T^*M which are polynomial in the cotangent direction. It is a Poisson algebra under the natural symplectic structure of T^*M. This Poisson algebra is the semiclassical limit of the algebra of differential operators on M, as we shall see in the next example.

Example 0.10.2 (Differential operators on commutative algebras)**.** Let A be a commutative unital algebra. We define an algebra $\mathcal{D}(A) \subset \mathrm{End}_{\mathbb{C}}(A)$ inductively as follows. Let

$$\mathcal{D}^0(A) = A = \mathrm{End}_A(A) \subset \mathrm{End}_{\mathbb{C}}(A)$$

denote the set of differential operators of order zero on A, i.e., A-linear maps from $A \to A$. Assuming $\mathcal{D}^k(A)$ has been defined for $0 \leq k < n$, we let $\mathcal{D}^n(A)$ be the set of all operators D in $\mathrm{End}_{\mathbb{C}}(A)$ such that for any $a \in A$, $[D, a] \in \mathcal{D}^{n-1}(A)$. The set

$$\mathcal{D}(A) = \bigcup_{n \geq 0} \mathcal{D}^n(A)$$

is a subalgebra of $\mathrm{End}_{\mathbb{C}}(A)$, called the *algebra of differential operators* on A. It is an almost commutative algebra under the filtration given by subspaces $\mathcal{D}^n(A)$, $n \geq 0$. Elements of $\mathcal{D}^n(A)$ are called differential operators of order n. For example, a linear map $D\colon A \to A$ is a differential operator of order one if and only if it is of the form $D = \delta + a$, where δ is a derivation on A and $a \in A$.

For general A, the semiclassical limit $\mathrm{Gr}(\mathcal{D}(A))$ and its Poisson structure are not easily identified except for coordinate rings of smooth affine varieties or algebras of smooth functions on a manifold. In this case a differential operator D of order k is locally given by

$$D = \sum_{|I| \leq k} a_I(x) \partial^I,$$

where $I = (i_1, \ldots, i_n)$ is a multi-index, $\partial^I = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_n}$ is a mixed partial derivative, and n is the dimension of the manifold. This expression depends on the local coordinates but its leading terms of total degree n have an invariant meaning provided that we replace ∂_i with $\xi_i \in T^*M$. For $\xi \in T_x^*M$, let

$$\sigma_p(D)(x, \xi) := \sum_{|I|=k} a_I(x)\xi^I.$$

Then the function $\sigma_p(D)\colon T^*M \to \mathbb{C}$, called the *principal symbol* of D, is invariantly defined and belongs to $C^\infty_{\text{poly}}(T^*M)$. The algebra $C^\infty_{\text{poly}}(T^*M)$ inherits a canonical Poisson structure as a subalgebra of the Poisson algebra $C^\infty(T^*M)$ and we have the following

Proposition 0.10.1. *The principal symbol map induces an isomorphism of Poisson algebras*

$$\sigma_p\colon \operatorname{Gr}\mathcal{D}(C^\infty(M)) \xrightarrow{\ \sim\ } C^\infty_{\text{poly}}(T^*M).$$

Example 0.10.3 (Weyl algebra)**.** Let $A_1 := \mathcal{D}\,\mathbb{C}[X]$ be the *Weyl algebra* of differential operators on the line. Alternatively, A_1 can be described as the unital complex algebra defined by generators x and p with

$$px - xp = 1.$$

The map $x \mapsto x$, $p \mapsto \frac{d}{dx}$ defines the isomorphism. Physicists prefer to write the defining relation as the *canonical commutation relation* $pq - qp = \frac{h}{2\pi i}1$, where h is Planck's constant and p and q represent momentum and position operators. This is not without merit because we can then let $h \to 0$ and obtain the commutative algebra of polynomials in p and q as the semiclassical limit. Also, i is necessary if we want to consider p and q as selfadjoint operators (why?). Then one can use the representation $q \mapsto x$, $p \mapsto \frac{h}{2\pi i}\frac{d}{dx}$.

Any element of A_1 has a unique expression as a differential operator with polynomial coefficients $\sum a_i(x)\frac{d^i}{dx^i}$ where the standard filtration is by degree of the differential operator. The principal symbol map

$$\sigma_p\Big(\sum_{i=0}^n a_i(x)\frac{d^i}{dx^i}\Big) = a_n(x)y^n.$$

defines an algebra isomorphism $\operatorname{Gr}(A_1) \simeq \mathbb{C}[x, y]$. The induced Poisson bracket on $\mathbb{C}[x, y]$ is the classical Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

In principle, the Weyl algebra A_n is the algebra of differential operators on $\mathbb{C}[x_1, \ldots, x_n]$. Alternatively, it can be defined as the universal algebra defined by $2n$ generators $x_1, \ldots, x_n, p_1, \ldots, p_n$ with

$$[p_i, x_i] = \delta_{ij} \quad \text{and} \quad [p_i, p_j] = [x_i, x_j] = 0$$

for all i, j. Notice that $A_n \simeq A_1 \otimes \cdots \otimes A_1$ (n factors). A lot is known about Weyl algebras and a lot remains to be known, including the Dixmier conjecture about the automorphisms of A_n. The Hochschild and cyclic cohomology of A_n are computed in [36] (cf. also [56]).

Example 0.10.4 (Universal enveloping algebras)**.** Let $U(\mathfrak{g})$ denote the *enveloping algebra* of a Lie algebra $\mathfrak{g}$. By definition, $U(\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$. For $p \geq 0$, let $F^p(U(\mathfrak{g}))$ be the subspace generated by tensors of degree at most p. This turns $U(\mathfrak{g})$ into a filtered algebra and the Poincaré–Birkhoff–Witt theorem asserts that its associated graded algebra is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$. The algebra isomorphism is induced by the *symmetrization map* $s\colon S(\mathfrak{g}) \to \mathrm{Gr}(U(\mathfrak{g}))$, defined by

$$s(X_1 X_2 \dots X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} X_{\sigma(1)} \dots X_{\sigma(p)}.$$

Note that $S(\mathfrak{g})$ is the algebra of polynomial functions on the dual space $\mathfrak{g}^*$, which is a Poisson manifold under the bracket

$$\{f, g\}(X) = [Df(X), \;\; Dg(X)]$$

for all $f, g \in C^\infty(\mathfrak{g}^*)$ and $X \in \mathfrak{g}^*$. Here we have used the canonical isomorphism $\mathfrak{g} \simeq \mathfrak{g}^{**}$, to regard the differential $Df(X) \in \mathfrak{g}^{**}$ as an element of $\mathfrak{g}$. The induced Poisson structure on polynomial functions coincides with the Poisson structure in $\mathrm{Gr}(U(\mathfrak{g}))$.

Example 0.10.5 (Algebra of formal pseudodifferential operators on the circle)**.** This algebra is obtained by formally inverting the differentiation operator $\partial := \frac{d}{dx}$ and then completing the resulting algebra. A formal pseudodifferential operator on the circle is an expression of the form $\sum_{-\infty}^{n} a_i(x)\partial^i$, where each $a_i(x)$ is a Laurent polynomial. The multiplication is uniquely defined by the rules $\partial x - x\partial = 1$ and $\partial\partial^{-1} = \partial^{-1}\partial = 1$. We denote the resulting algebra by Ψ_1. The *Adler–Manin trace* on Ψ_1 [58], also called the *noncommutative residue*, is defined by

$$\mathrm{Tr}\Big(\sum_{-\infty}^{n} a_i(x)\partial^i\Big) = \mathrm{Res}(a_{-1}(x);\, 0) = \frac{1}{2\pi i}\int_{S^1} a_{-1}(x).$$

This is a trace on Ψ_1. In fact one can show that $\Psi_1/[\Psi_1, \Psi_1]$ is 1-dimensional which means that any trace on Ψ_1 is a multiple of Tr. Notice that for the Weyl algebra A_1 we have $[A_1, A_1] = A_1$.

Another interesting difference between Ψ_1 and A_1 is that Ψ_1 admits non-inner derivations (see exercise below). The algebra Ψ_1 has a nice generalization to algebras of pseudodifferential operators in higher dimensions. The appropriate extension of the above trace is the *noncommutative residue* of Wodzicki (cf. [68]; see also [21] for relations with the Dixmier trace and its role in noncommutative Riemannian geometry).

So far in this section we saw at least one way to formalize the idea of quantization through the notion of an almost commutative algebra and its semiclassical limit which is a Poisson algebra. A closely related notion is *formal deformation quantization*, or *star products*, going back to [5], [62]. It is also closely related to the theory of deformations of associative algebras developed originally by Gerstenhaber, as we recall now.

Let A be an algebra, which may be noncommutative, over $\mathbb{C}$, and let $A[[h]]$ be the algebra of formal power series over A. A *formal deformation* of A is an associative $\mathbb{C}[[h]]$-linear multiplication

$$*_h : A[[h]] \otimes A[[h]] \to A[[h]]$$

such that $*_0$ is the original multiplication. Writing

$$a *_h b = B_0(a, b) + hB_1(a, b) + h^2 B_2(a, b) + \cdots ,$$

where $B_i : A \otimes A \to A$ are Hochschild 2-cochains on A with values in A, we see that the initial value condition on $*_h$ is equivalent to $B_0(a, b) = ab$ for all $a, b \in A$. Let us define a bracket $\{\ ,\ \}$ on A by

$$\{a,\, b\} = B_1(a, b) - B_1(b, a)$$

or, equivalently, but more suggestively, by

$$\{a,\, b\} := \mathrm{Lim}\, \frac{a *_h b - b *_h a}{h} \quad \text{as } h \to 0.$$

Using the associativity of the star product $a *_h (b *_h c) = (a *_h b) *_h c$, it is easy to check that B_1 is a Hochschild 2-cocycle for the Hochschild cohomology of A with coefficients in A, i.e., it satisfies the relation

$$aB_1(b, c) - B_1(ab, c) + B_1(a, bc) - B_1(a, b)c = 0$$

for all a, b, c in A. Clearly the bracket $\{\ ,\ \}$ satisfies the Jacobi identity. In short, $(A, \{\ ,\ \})$ is an example of what is sometimes called a *noncommutative Poisson algebra*. If A is a commutative algebra, then it is easy to see that it is indeed a Poisson algebra in the sense of Definition 0.10.3.

The bracket $\{\ ,\ \}$ can be regarded as the *infinitesimal direction* of the deformation, and the deformation problem for a commutative Poisson algebra amounts to finding higher order terms B_i, $i \geq 2$, given B_0 and B_1.

The associativity condition on $*_h$ is equivalent to an infinite system of equations involving the cochains B_i that we derive now. They are given by

$$B_0 \circ B_n + B_1 \circ B_{n-1} + \cdots + B_n \circ B_0 = 0 \quad \text{for all } n \geq 0,$$

or, equivalently,

$$\sum_{i=1}^{n-1} B_i \circ B_{n-i} = \delta B_n. \tag{14}$$

Here, the *Gerstenhaber* $\circ$*-product* of 2-cochains f, $g\colon A\otimes A\to A$ is defined as the 3-cochain

$$f\circ g(a,b,c)=f(g(a,b),c)-f(a,g(b,c)).$$

Notice that a 2-cochain f defines an associative product if and only if $f\circ f=0$. Also notice that the Hochschild coboundary δf can be written as

$$\delta f=-m\circ f-f\circ m,$$

where $m\colon A\otimes A\to A$ is the multiplication of A. These observations lead to the associativity equations (14).

To solve these equations starting with $B_0=m$, by antisymmetrizing we can always assume that B_1 is antisymmetric and hence we can assume $B_1=\frac{1}{2}\{\,,\,\}$. Assume $B_0,B_1,\ldots,B_n$ have been found so that (14) holds. Then one can show that $\sum_{i=1}^{n}B_i\circ B_{n-i}$ is a cocycle. Thus we can find a B_{n+1} satisfying (14) if and only if this cocycle is a coboundary, i.e., its class in $H^3(A,A)$ should vanish. The upshot is that the third Hochschild cohomology $H^3(A,A)$ is the *space of obstructions* for the deformation quantization problem. In particular if $H^3(A,A)=0$ then any Poisson bracket on A can be deformed. Notice, however, that this is only a sufficient condition and is by no means necessary, as will be shown below.

In the most interesting examples, e.g. for $A=C^\infty(M)$, $H^3(A,A)\neq 0$. To see this consider the differential graded Lie algebra $(C^*(A,\,A),\,[\,,\,],\,\delta)$ of continuous Hochschild cochains on A, and the differential graded Lie algebra, with zero differential, $(\bigwedge(TM),\,[\,,\,],\,0)$ of polyvector fields on M. The bracket in the first is the Gerstenhaber bracket and in the second is the Schouten bracket of polyvector fields. By a theorem of Connes (see the resolution in Lemma 44 in [19]), we know that the *antisymmetrization map*

$$\alpha\colon\big(C^\infty\big(\textstyle\bigwedge TM),\,0\big)\to\big(C^*(A,\,A),\,\delta\big)$$

sending a polyvector field $X_1\wedge\cdots\wedge X_k$ to the functional φ defined by

$$\varphi(f^1,\ldots,f^k)=df^1(X_1)df^2(X_2)\ldots df^k(X_k)$$

is a quasi-isomorphism of differential graded algebras. In particular, it induces an isomorphism of graded commutative algebras

$$\bigoplus_k H^k(A,A)\simeq\bigoplus_k C^\infty\big(\textstyle\bigwedge^k TM\big).$$

The map α, however, is not a morphism of Lie algebras and that is where the real difficulty of deforming a Poisson structure is hidden. The *formality theorem* of M. Kontsevich [53] states that as a differential graded Lie algebra, $(C^*(A,A),\delta,[\,,\,])$ is *formal* in the sense that it is quasi-isomorphic to its cohomology. Equivalently, it means that one can perturb the map α, by adding an infinite number of terms, to a morphism of L_∞-algebras. This shows that the original deformation problem of Poisson structures can be transferred to $(C^\infty(\bigwedge TM),\,0)$

where it is unobstructed since the differential in the latter DGL is zero. Later in this section we shall give a couple of simple examples where deformations can be explicitly constructed.

There is a much deeper structure hidden in the deformation complex of an associative $(C^*(A,A),\,\delta)$ than first meets the eye and we can only barely scratch the surface here. The first piece of structure is the cup product. Let $C^* = C^*(A,A)$. The *cup product* $\smile\colon C^p \times C^q \to C^{p+q}$ is defined by

$$(f \smile g)(a^1, \ldots, a^{p+q}) = f(a^1, \ldots, a^p)g(a^{p+1}, \ldots, a^{p+q}).$$

Notice that $\smile$ is associative and one checks that this product is compatible with the differential δ and hence induces an associative graded product on $H^*(A,\,A)$. What is not so obvious however is that this product is graded commutative for any algebra A [37].

The second piece of structure on $(C^*(A,A),\,\delta)$ is a graded Lie bracket. It is based on the Gerstenhaber circle product $\circ\colon C^p \times C^q \to C^{p+q-1}$ defined by

$$\begin{aligned}(f \circ g)&(a_1, \ldots, a_{p+q-1})\\ &= \sum_{i=1}^{p-1}(-1)^{|g|(|f|+i-1)} f(a^1, \ldots, g(a^i, \ldots, a^{i+p}), \ldots, a^{p+q-1}).\end{aligned}$$

Notice that $\circ$ is not an associative product. Nevertheless one can show that [37] the corresponding graded bracket $[\;,\;]\colon C^p \times C^q \to C^{p+q-1}$

$$[f,\,g] = f \circ g - (-1)^{(p-1)(q-1)} g \circ f$$

defines a graded Lie algebra structure on the deformation cohomology $H^*(A,\,A)$. Notice that the Lie algebra grading is now shifted by one.

What is most interesting is that the cup product and the Lie algebra structure are compatible in the sense that $[\;,\;]$ is a graded derivation for the cup product; or in short, $(H^*(A,\,A), \smile, [\;,\;])$ is a *graded Poisson algebra*.

We give a couple of examples where deformations can be explicitly constructed.

Example 0.10.6. The simplest non-trivial Poisson manifold is the dual $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$. Let $U_h(\mathfrak{g}) = T(\mathfrak{g})/I$, where the ideal I is generated by

$$x \otimes y - y \otimes x - h[x,y], \quad x, y \in \mathfrak{g}.$$

This is simply the enveloping algebra of the rescaled bracket $h[-,-]$. By the Poincaré–Birkhoff–Witt theorem, the antisymmetrization map $\alpha_h\colon S(\mathfrak{g}) \to U_h(\mathfrak{g})$ is a linear isomorphism. We can define a $*$-product on $S(\mathfrak{g})$ by

$$f *_h g = \alpha_h^{-1}(\alpha_h(f)\alpha_h(g)) = \sum_{n=0}^{\infty} h^n B_n(f,g).$$

With some work one can show that the B_n are bidifferential operators and hence the formula extends to a $*$-product on $C^\infty(\mathfrak{g}^*)$.

Example 0.10.7 (Weyl–Moyal quantization)**.** Consider the algebra generated by x and y with relation $xy - yx = \frac{h}{i}1$. Let f, g be polynomials in x and y. Iterated application of the Leibniz rule gives the formula for the product

$$f *_h g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-ih}{2} \right)^n B_n(f, g),$$

where $B_0(f, g) = fg$, $B_1(f, g) = \{f, g\}$ is the standard Poisson bracket, and for $n \geq 2$,

$$B_n(f,\, g) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\partial_x^k \partial_y^{n-k} f)(\partial_x^{n-k} \partial_y^k g).$$

Notice that this formula makes sense for $f,\, g \in C^\infty(\mathbb{R}^2)$ and defines a deformation of this algebra with its standard Poisson structure. This can be extended to arbitrary *constant* Poisson structures on $\mathbb{R}^2$,

$$\{f,\, g\} = \sum \pi^{ij} \partial_i f \, \partial_j g.$$

The Weyl–Moyal $*$ product is then given by

$$f *_h g = \exp\left(-i\frac{h}{2} \sum \pi^{ij} \partial_i \wedge \partial_j \right)(f,\, g).$$

Exercise 0.10.1. Show that the Weyl algebra A_1 is a simple algebra, i.e., it has no non-trivial two-sided ideals; prove the same for A_n. In a previous exercise we asked to show that any derivation of A_1 is inner. Is it true that any automorphism of A_1 is inner?

Exercise 0.10.2. In Example 0.10.3 show that there is no linear map $q\colon \mathbb{C}[x, y] \to A_1$ such that $q(1) = 1$ and $q\{f,\, g\} = [q(f),\, q(g)]$ for all f and g. This is an important special case of the Groenewold–van Hove no-go theorem ([40]).

Exercise 0.10.3. Let $A = \mathbb{C}[x]/(x^2)$ be the algebra of dual numbers. It is a non-smooth algebra. Describe its algebra of differential operators.

Exercise 0.10.4. Unlike the algebra of differential operators, the algebra of pseudodifferential operators Ψ_1 admits non-inner derivations. Clearly $\log \partial := -\sum_1^\infty \frac{(1-\partial)^n}{n} \notin \Psi_1$, but show that for any $a \in \Psi_1$, we have $[\log \partial,\, a] \in \Psi_1$ and therefore the map

$$a \mapsto \delta(a) := [\log\, \partial,\, a]$$

defines a non-inner derivation of Ψ_1. The corresponding Lie algebra 2-cocycle

$$\varphi(a, b) = \mathrm{Tr}(a[\log\, \partial,\, b])$$

is the Radul cocycle [54].

0.11 Topological algebras

For applications of Hochschild and cyclic cohomology to noncommutative geometry, it is crucial to consider topological algebras, topological bimodules, topological resolutions, and continuous cochains and chains. For example, while the algebraic Hochschild groups of the algebra of smooth functions on a smooth manifold are not known, and perhaps are hopeless to compute, its continuous Hochschild (co)homology as a topological algebra can be computed as we recall in Example 0.9.3 below. We shall give only a brief outline of the definitions and refer the reader to [19], [21] for more details. A good reference for locally convex topological vector spaces and topological tensor products is the book of Treves on the subject.

There is no difficulty in defining *continuous* analogues of Hochschild and cyclic cohomology groups for Banach algebras. One simply replaces bimodules by Banach bimodules, that is a bimodule which is also a Banach space where the left and right module actions are bounded operators, and cochains by continuous cochains. Since the multiplication of a Banach algebra is a continuous operation, all operators including the Hochschild boundary and the cyclic operators extend to this continuous setting. The resulting Hochschild and cyclic theory for C^*-algebras, however, is hardly useful and tends to vanish in many interesting examples. This is hardly surprising since the definition of any Hochschild or cyclic cocycle of dimension bigger than zero involves differentiating the elements of the algebra in one way or another. (See Exercise 0.8.3 or, more generally, the Remark below.) This is in sharp contrast with topological K-theory where the right setting, e.g. for Bott periodicity to hold, is the setting of Banach or C^*-algebras.

Remark 3. By combining results of Connes [15] and Haagerup [41], we know that a C^*-algebra is *amenable* if and only if it is *nuclear*. Amenability refers to the property that for all $n \geq 1$,

$$H^n_{\text{cont}}(A, M^*) = 0$$

for an arbitrary Banach dual bimodule M^*. In particular, for any nuclear C^*-algebra
$HH^n_{\text{cont}}(A) = H^n_{\text{cont}}(A, A^*) = 0$ for all $n \geq 1$. Using Connes' long exact sequence (see Section 0.14), we obtain, for any nuclear C^*-algebra A, the vanishing results

$$HC^{2n}_{\text{cont}}(A) = A^* \quad \text{and} \quad HC^{2n+1}_{\text{cont}}(A) = 0$$

for all $n \geq 0$. Nuclear C^*-algebras form a large class which includes commutative algebras, the algebra of compact operators, and reduced group C^*-algebras of amenable groups [8].

The right class of topological algebras for Hochschild and cyclic cohomology turns out to be the class of *locally convex algebras* [19]. An algebra A which is simultaneously a locally convex topological vector space is called a locally convex algebra if its multiplication map $A \otimes A \to A$ is (jointly) continuous. That is, for any continuous seminorm p on A there is a continuous seminorm p' on A such that $p(ab) \leq p'(ab)$ for all a, b in A.

We should mention that there are topological algebras with a locally convex topology for which the multiplication map is only *separately continuous*. But we do not dwell on this more general class in this book as they appear rarely in applications. This distinction between separate and joint continuity of the multiplication map disappears for the class of Fréchet algebras. By definition, a locally convex algebra is called a *Fréchet algebra* if its topology is metrizable and complete. Many examples of 'smooth noncommutative spaces' that one encounters in noncommutative geometry are in fact Fréchet algebras.

Example 0.11.1. Basic examples of Fréchet algebras include the algebra $A = C^\infty(M)$ of smooth functions on a closed smooth manifold and the smooth noncommutative tori $\mathcal{A}_\theta$ and their higher dimensional analogues. We start with a simple down to earth example where $A = C^\infty(S^1)$. We consider the elements of A as smooth periodic functions of period one on the line. Its topology is defined by the sequence of norms

$$p_n(f) = \sup \|f^{(k)}\|_\infty, \quad 0 \leq k \leq n,$$

where $f^{(k)}$ is the k-th derivative of f and $\| \|_\infty$ is the sup norm. We can equivalently use the sequence of norms

$$q_n(f) = \sum_{i=0}^{n} \frac{1}{k!} \|f^{(k)}\|_\infty.$$

Notice that q_n's are *submultiplicative*, that is $q_n(fg) \leq q_n(f)q_n(g)$. Locally convex algebras whose topology is induced by a family of submultiplicative seminorms are known to be projective limits of Banach algebras. This is the case in all examples in this section.

In general, the topology of $C^\infty(M)$ is defined by the sequence of seminorms

$$\|f\|_n = \sup |\partial^\alpha f|, \quad |\alpha| \leq n,$$

where the supremum is over a fixed, finite, coordinate cover for M. The Leibniz rule for derivatives of products shows that the multiplication map is indeed jointly continuous. See the Exercise 0.11.1 for the topology of $\mathcal{A}_\theta$.

Given locally convex topological vector spaces V_1 and V_2, their *projective tensor product* is a locally convex space $V_1 \hat{\otimes} V_2$ together with a universal jointly continuous bilinear map $V_1 \otimes V_2 \to V_1 \hat{\otimes} V_2$ (cf. [39]). That is, for any locally convex space W, we have a natural isomorphism between jointly continuous bilinear maps $V_1 \times V_2 \to W$ and continuous linear maps $V_1 \hat{\otimes} V_2 \to W$. Explicitly, the topology of $V_1 \hat{\otimes} V_2$ is defined by the family of seminorms $p \hat{\otimes} q$, where p, q are continuous seminoms on V_1 and V_2 respectively, and

$$p \hat{\otimes} q(t) := \inf \left\{ \textstyle\sum_i^n p(a_i)q(b_i);\ t = \sum_i^n a_i \otimes b_i,\ a_i \in V_1,\ b_i \in V_2 \right\}.$$

Then $V_1 \hat{\otimes} V_2$ is defined as the completion of $V_1 \otimes V_2$ under the above family of seminorms.

One of the nice properties of the topology of $C^\infty(M)$ is that it is *nuclear* (see [39]). In particular for any other smooth compact manifold N, the natural map

$$C^\infty(M) \hat{\otimes} C^\infty(N) \to C^\infty(M \times N)$$

is an isomorphism of topological algebras. This plays an important role in computing the continuous Hochschild cohomology of $C^\infty(M)$.

Let A be a locally convex topological algebra. A topological left A-module is a locally convex vector space M endowed with a continuous left A-module action $A \hat{\otimes} M \to M$. A *topological free* left A-module is a module of the type $M = A \hat{\otimes} V$ where V is a locally convex space. A *topological projective module* is a topological module which is a direct summand in a free topological module.

Given a locally convex algebra A, let

$$C^n_{\text{cont}}(A) = \text{Hom}_{\text{cont}}(A^{\hat{\otimes} n}, \mathbb{C})$$

be the space of continuous $(n+1)$-linear functionals on A. All the algebraic definitions and results of this chapter extend to this topological setting. In particular one defines the *continuous Hochschild* and *cyclic cohomology* groups of a locally convex algebra. One must be careful, however, in dealing with resolutions. The right class of topological projective (in particular free) resolutions are those resolutions that admit a continuous linear splitting. This extra condition is needed when one wants to prove comparison theorems for resolutions and, eventually, independence of cohomology from resolutions. We shall not go into details here since this is very well explained in [19].

Exercise 0.11.1. The sequence of norms

$$p_k(a) = \underset{m,n}{\text{Sup}} \{(1 + |n| + |m|)^k |a_{mn}|\}$$

defines a locally convex topology on the smooth noncommutative torus $\mathcal{A}_\theta$. Show that the multiplication of $\mathcal{A}_\theta$ is continuous in this topology.

0.12 Hochschild (co)homology computations

We give a few examples of Hochschild (co)homology computations. In particular we shall see that group (co)homology and Lie algebra (co)homology are instances of Hochschild (co)homology. We start by recalling the classical results of Hochschild–Kostant–Rosenberg [47] and Connes [19] which identifies the Hochschild homology of *smooth commutative algebras* with the algebra of differential forms. By a smooth commutative algebra we mean either the topological algebra $A = C^\infty(M)$ of smooth complex-valued functions on a closed smooth manifold M, or the algebra $A = \mathcal{O}[X]$ of regular function on a smooth affine algebraic variety X. We start with the latter case.

Example 0.12.1 (Smooth commutative algebras)**.** Algebras of regular functions on a smooth affine variety can be characterized abstractly through various equivalent conditions (cf. Proposition 3.4.2 in [56]). For example, one knows that a finitely generated commutative algebra A is smooth if and only if it has the *lifting property* with respect to *nilpotent extensions.* More precisely, A is smooth if and only if for any pair (C, I), where C is a commutative algebra and I is an ideal such that $I^2 = 0$, the map

$$\mathrm{Hom}_{\mathrm{alg}}(A,\, C) \to \mathrm{Hom}_{\mathrm{alg}}(A,\, C/I)$$

is surjective. Examples of smooth algebras include polynomial algebras $\mathbb{C}[x_1, \dots, x_n]$, algebras of Laurent polynomials $\mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$, and coordinate rings of affine algebraic groups. The algebra $\mathbb{C}[x, y]/(xy)$ is not smooth.

We recall the definition of the *algebraic de Rham complex* of a commutative, not necessarily smooth, algebra A. The module of 1-forms, or *Kähler differentials*, over A, denoted by Ω^1_A, is by definition a left A-module Ω^1_A endowed with a *universal derivation*

$$d\colon A \to \Omega^1_A.$$

This means that any other derivation $\delta\colon A \to M$ into a left A-module M uniquely factorizes through d. One usually defines $\Omega^1_A := I/I^2$ where I is the kernel of the multiplication map $A \otimes A \to A$. Note that since A is commutative this map is an algebra homomorphism and I is an ideal. The left multiplication defines a left A-module structure on Ω^1_A. The derivation d is defined by

$$d(a) = a \otimes 1 - 1 \otimes a \mod (I^2).$$

Checking its universal property is straightforward. One defines the space of n-forms on A as the n-th exterior power of the A-module Ω^1_A:

$$\Omega^n_A := \textstyle\bigwedge^n_A \Omega^1_A,$$

where the exterior product is over A. There is a unique extension of d to a graded derivation of degree one,

$$d\colon \Omega^*_A \to \Omega^{*+1}_A.$$

It satisfies $d^2 = 0$. The *algebraic de Rham cohomology* of A is the cohomology of the complex $(\Omega^*_A,\, d)$. For some examples of this construction see exercises at the end of this section.

Let us compare the complex of differential forms with trivial differential $(\Omega^*_A,\, 0)$, with the Hochschild complex of A with coefficients in A, $(C_*(A),\, b)$. Consider the *antisymmetrization map*

$$\varepsilon_n\colon \Omega^n_A \to A^{\otimes(n+1)}, \quad n = 0, 1, 2, \dots,$$

$$\varepsilon_n(a_0 da_1 \wedge \cdots \wedge da_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

where S_n denotes the symmetric group on n letters. We also have maps

$$\mu_n\colon A^{\otimes(n+1)} \to \Omega^n_A, \quad n = 0, 1, \ldots,$$

$$\mu_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 da_1 \wedge \cdots \wedge da_n.$$

One checks that both maps are morphisms of complexes, i.e.,

$$b \circ \varepsilon_n = 0 \quad \text{and} \quad \mu_n \circ b = 0.$$

Moreover, one has

$$\mu_n \circ \varepsilon_n = n!\,\mathrm{id}.$$

Assuming the ground field has characteristic zero, it follows that the antisymmetrization map induces an inclusion

$$\Omega^n_A \hookrightarrow HH_n(A), \quad n = 0, 1, 2, \ldots. \tag{15}$$

In particular, for any commutative algebra A over a field of characteristic zero, the space of differential n-forms on A is always a direct summand of the Hochschild homology group $HH_n(A)$.

This map, however, need not be surjective in general (cf. Exercises below). This has to do with the *singularity* of the underlying geometric space represented by A. The Hochschild–Kostant–Rosenberg theorem [47] states that if A is the algebra of regular functions on a smooth affine variety, then the above map is an isomorphism. Notice that we have verified this fact for polynomial algebras in Example 0.9.2.

Example 0.12.2 (Algebras of smooth functions)**.** This is a continuation of Example 0.8.2. Let M be a smooth closed manifold and $A = C^\infty(M)$ the algebra of smooth complex-valued functions on M. It is a locally convex (in fact, Fréchet) topological algebra as we explained in Section 0.11. Let $\Omega^n M$ (resp. $\Omega_n M$) denote the space of n-forms (resp. n-currents) on M. Consider the map

$$\Omega_n M \to C^n_{\text{cont}}(C^\infty(M)), \quad C \mapsto \varphi_C,$$

where

$$\varphi_C(f_0, f_1, \ldots, f_n) := \langle C,\, f_0 df_1 \ldots df_n\rangle.$$

It is easily checked that this map defines a morphism of complexes

$$(\Omega_* M,\, 0) \to (C^*_{\text{cont}}(C^\infty(M)),\, b).$$

In [19], using an explicit resolution, Connes shows that the induced map on cohomologies is an isomorphism. Thus we have a natural isomorphism between space of de Rham currents on M and (continuous) Hochschild cohomology of $C^\infty(M)$:

$$\boxed{\Omega_i M \simeq HH^i_{\text{cont}}(C^\infty(M)), \quad i = 0, 1, \ldots} \tag{16}$$

Without going into details, we shall briefly indicate the resolution introduced in [19]. Let $\bigwedge^k T^*_{\mathbb{C}}M$ denote the bundle of complexified k-forms on M and E_k be its pullback under the projection $\mathrm{pr}_2\colon M \times M \to M$. Let X be a vector field on $M^2 = M \times M$ such that in a neighborhood of the diagonal $\Delta(M) \subset M \times M$ and in a local geodesic coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ it looks like

$$X = \sum_{i=1}^{n} (x_i - y_i) \frac{\partial}{\partial y_i}.$$

We assume that away from the diagonal X is nowhere zero. Such an X can always be found, provided that M admits a nowhere zero vector field. The latter condition is clearly equivalent to vanishing of the Euler characteristic of M. By replacing M by $M \times S^1$ the general case can be reduced to this special case.

The following is then shown to be a continuous projective resolution of $C^\infty(M)$ as a $C^\infty(M \times M)$-module [19]:

$$C^\infty(M) \xleftarrow{\varepsilon} C^\infty(M^2) \xleftarrow{i_X} C^\infty(M^2, E_1)$$
$$\xleftarrow{i_X} C^\infty(M^2, E_2) \xleftarrow{i_X} \cdots \xleftarrow{i_X} C^\infty(M^2, E_n) \leftarrow 0,$$

where i_X is interior multiplication by X. By applying the Hom functor $\mathrm{Hom}_{A \otimes A}(-, A^*)$ we obtain a complex with zero differentials

$$\Omega_0 M \xrightarrow{0} \Omega_1 M \xrightarrow{0} \cdots \xrightarrow{0} \Omega_n M \xrightarrow{0} 0,$$

and hence the isomorphism (16).

The analogous result for Hochschild homology uses the map

$$C^\infty(M) \hat{\otimes} \cdots \hat{\otimes} C^\infty(M) \to \Omega^n M$$

defined by

$$f_0 \otimes \cdots \otimes f_n \mapsto f_0 df_1 \ldots df_n.$$

It is easy to check that this defines a morphism of complexes

$$C_*^{\mathrm{cont}}(C^\infty(M), b) \to (\Omega^* M, 0).$$

Using the above resolution and by essentially the same argument, one shows that the induced map on homologies is an isomorphism between continuous Hochschild homology of $C^\infty(M)$ and differential forms on M:

$$\boxed{HH_i^{\mathrm{cont}}(C^\infty(M)) \simeq \Omega^i M, \quad i = 0, 1, \ldots}$$

Example 0.12.3 (Group algebras)**.** It is clear from the original definitions that group (co)homology is an example of Hochschild (co)homology. Let G be a group and M be a left G-module. The standard complex for computing the cohomology of G with coefficients in M is the complex (cf. [14], [56])

$$M \xrightarrow{\delta} C^1(G, M) \xrightarrow{\delta} C^2(G, M) \xrightarrow{\delta} \cdots,$$

where

$$C^n(G,M) = \{f\colon G^n \to M\},$$

and the differential δ is defined by

$$(\delta m)(g) = gm - m,$$

$$\delta f(g_1,\ldots,g_{n+1}) = g_1 f(g_2,\ldots,g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1,\ldots,g_i g_{i+1},\ldots,g_{n+1}) + (-1)^{n+1} f(g_1,g_2,\ldots,g_n).$$

Let $A = \mathbb{C}G$ denote the group algebra of G over the complex numbers. Then M is a $\mathbb{C}G$-bimodule via the actions

$$g \cdot m = g(m), \quad m \cdot g = m$$

for all g in G and m in M. It is clear that for all n,

$$C^n(\mathbb{C}G,\, M) \simeq C^n(G,\, M),$$

and the isomorphism preserves the differentials. It follows that the group cohomology of G with coefficients in M coincides with the Hochschild cohomology of $\mathbb{C}G$ with coefficients in M:

$$H^n(\mathbb{C}G,\, M) \simeq H^n(G,\, M).$$

Conversely, it is easy to see that the Hochschild cohomology of $\mathbb{C}G$ with coefficients in a bimodule M reduces to group cohomology. Let $M^{\mathrm{ad}} = M$ as a vector space and define a left G-action on M^{ad} by

$$g \cdot m = gmg^{-1}.$$

Define a linear isomorphism $i\colon C^n(G,\, M^{\mathrm{ad}}) \to C^n(\mathbb{C}G,\, M)$ by

$$(if)(g_1,\ldots,g_n) = f(g_1,\ldots,g_n) g_1 g_2 \ldots g_n.$$

It can be checked that i commutes with differentials and hence is an isomorphism of complexes (MacLane isomorphism)

$$C^*(G,\, M^{\mathrm{ad}}) \xrightarrow{\ \simeq\ } C^*(\mathbb{C}G,\, M).$$

Of course, there is a similar result for homology.

Of particular importance is an understanding of $HH_*(\mathbb{C}G) = H_*(\mathbb{C}G,\, \mathbb{C}G) = H_*(G,\, \mathbb{C}G)$, i.e., when $M = \mathbb{C}G$ and G is acting by *conjugation*. By a theorem of Burghelea [11], the Hochschild and cyclic homology groups of $\mathbb{C}G$ decompose over the set of conjugacy classes of G where each summand is the group homology (with trivial coefficients) of a group associated to the conjugacy class. We recall this result for Hochschild homology here and later we shall discuss the corresponding result for cyclic homology.

The whole idea can be traced back to the following simple observation. Let $\tau\colon \mathbb{C}G \to \mathbb{C}$ be a trace on the group algebra. It is clear that τ is constant on each conjugacy class of G and, conversely, the characteristic function of each conjugacy class defines a trace on $\mathbb{C}G$. Thus we have

$$HH^0(\mathbb{C}G) = \prod_{\langle G\rangle} \mathbb{C},$$

where $\langle G\rangle$ denotes the set of conjugacy classes of G. Dually, for homology we have

$$HH_0(\mathbb{C}G) = \bigoplus_{\langle G\rangle} \mathbb{C}.$$

We focus on homology and shall extend the above observation to higher dimensions. Dual cohomological versions are straightforward.

Clearly we have

$$(\mathbb{C}G)^{\otimes(n+1)} = \mathbb{C}G^{n+1}.$$

For each conjugacy class $c \in \langle G\rangle$, let $B_n(G,\, c)$ be the linear span of all $(n+1)$-tuples $(g_0, g_1, \ldots, g_n) \in G^{n+1}$ such that

$$g_0 g_1 \ldots g_n \in c.$$

It is clear that $B_*(G,\, c)$ is invariant under the Hochschild differential b. We therefore have a decomposition of the Hochschild complex of $\mathbb{C}G$ into subcomplexes indexed by conjugacy classes:

$$C_*(\mathbb{C}G,\, \mathbb{C}G) = \bigoplus_{c\in\langle G\rangle} B_*(G,\, c).$$

Identifying the homology of the component corresponding to the conjugacy class of the identity is rather easy. For other components one must work harder. Let $c = \{e\}$ denote the conjugacy class of the identity element of G. The map $(g_0, g_1, \ldots, g_n) \mapsto (g_1, g_2, \ldots, g_n)$ is easily seen to define an isomorphism of vector spaces

$$B_n(G, \{e\}) \simeq \mathbb{C}G^n.$$

Moreover, under this map the Hochschild differential b goes over to the differential for the group homology of G with trivial coefficients. It follows that

$$H_*(B(G, \{e\})) \simeq H_*(G,\, \mathbb{C}).$$

Next we describe the Hochschild homology of other components. For an element $g \in G$, let

$$C_g = \{h \in G;\ hg = gh\}$$

denote the *centralizer* of g in G. Notice that the isomorphism class of this group depends only on the conjugacy class of g. One checks that the inclusion $i\colon C_n(C_g,\mathbb{C}) \to B_n(G,c)$ defined by

$$i(g_1,\dots,g_n) = ((g_1g_2\dots g_n)^{-1}g, g_1,\dots,g_{n-1})$$

is a chain map. One can in fact show, by an explicit chain homotopy, that i is a quasi-isomorphism. It therefore follows that, for each conjugacy class c and each $g \in c$, we have

$$H_*(B(G,\,c)) = H_*(C_g,\,\mathbb{C}).$$

Putting everything together this shows that the Hochschild homology of $\mathbb{C}G$ decomposes as a direct sum of group homologies of centralizers of conjugacy classes of G, a result due to Burghelea [11] (cf. also [56], for purely algebraic proofs):

$$\boxed{HH_*(\mathbb{C}G) \simeq \bigoplus_{\langle G\rangle} H_*(C_g)} \tag{17}$$

The corresponding dual statement for Hochschild cohomology reads as

$$HH^*(\mathbb{C}G) \simeq \prod_{\langle G\rangle} H^*(C_g).$$

Example 0.12.4 (Enveloping algebras)**.** We show that Lie algebra (co)homology is an example of Hochschild (co)homology, a result which goes back to Cartan–Eilenberg [14]. Let $\mathfrak{g}$ be a Lie algebra and M be a (left) $\mathfrak{g}$-module. This simply means that we have a Lie algebra morphism

$$\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(M).$$

The *Lie algebra homology* of $\mathfrak{g}$ with coefficients in M is, by definition, the homology of the *Chevalley–Eilenberg complex*

$$M \xleftarrow{\delta} M\otimes {\textstyle\bigwedge}^1\mathfrak{g} \xleftarrow{\delta} M\otimes {\textstyle\bigwedge}^2\mathfrak{g} \xleftarrow{\delta} M\otimes {\textstyle\bigwedge}^3\mathfrak{g} \xleftarrow{\delta} \cdots,$$

where the differential δ is defined by

$$\delta(m\otimes X) = X(m),$$

$$\begin{aligned}\delta(m\otimes X_1\wedge X_2\wedge\cdots\wedge X_n) = \\ \sum_{i<j}(-1)^{i+j} m\otimes[X_i,X_j]\wedge X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge\hat{X}_j\wedge\cdots\wedge X_n \\ +\sum_i(-1)^i X_i(m)\otimes X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge X_n.\end{aligned}$$

(One checks that $\delta^2=0$.)

Let $U(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$. Given a $U(\mathfrak{g})$-bimodule M, we define a left $\mathfrak{g}$-module M^{ad}, where $M^{\mathrm{ad}} = M$ and

$$X \cdot m := Xm - mX$$

for all $X \in \mathfrak{g}$ and $m \in M$. Define a map

$$\varepsilon\colon M^{\mathrm{ad}} \otimes \bigwedge^n \mathfrak{g} \to M \otimes U(\mathfrak{g})^{\otimes n}$$

from the Lie algebra complex to the Hochschild complex by

$$\varepsilon(m \otimes X_1 \wedge \cdots \wedge X_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) m \otimes X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}.$$

One checks that $\varepsilon\colon C^{\mathrm{Lie}}(\mathfrak{g}, M^{\mathrm{ad}}) \to C(U(\mathfrak{g}), M)$ is a chain map (prove this!). We claim that it is indeed a quasi-isomorphism, i.e., it induces an isomorphism between the corresponding homology groups:

$$H_*(\mathfrak{g}, M^{\mathrm{ad}}) \simeq H_*(U(\mathfrak{g}), M).$$

We refer to [14], [56] for its standard proof.

Example 0.12.5 (Morita invariance of Hochschild homology)**.** Let A and B be unital Morita equivalent algebras. Let X be an equivalence A–B-bimodule and Y be an inverse B–A-bimodule. Let M be an A-bimodule and $N = Y \otimes_A M \otimes_A X$ the corresponding B-bimodule. *Morita invariance* of Hochschild homology states that there is a natural isomorphism

$$H_n(A, M) \simeq H_n(B, N)$$

for all $n \geq 0$. A proof of this can be found in [56]. There is a similar result, with a similar proof, for cohomology. We sketch a proof of this result for the special case where $B = M_k(A)$ is the algebra of k by k matrices over A. The main idea is to introduce the *generalized trace map*.

Let M be an A-bimodule and $M_k(M)$ be the space of k by k matrices with coefficients in M. It is a bimodule over $M_k(A)$ in an obvious way. The *generalized trace map* is defined by

$$\mathrm{Tr}\colon C_n(M_k(A), M_k(M)) \to C_n(A, M),$$

$$\mathrm{Tr}(\alpha_0 \otimes m_0 \otimes \alpha_1 \otimes a_1 \otimes \cdots \otimes \alpha_n \otimes a_n) = \mathrm{tr}(\alpha_0 \alpha_1 \ldots \alpha_n) m_0 \otimes a_1 \otimes \cdots \otimes a_n,$$

where $\alpha_i \in M_k(\mathbb{C})$, $a_i \in A$, $m_0 \in M$, and $\mathrm{tr}\colon M_k(\mathbb{C}) \to \mathbb{C}$ is the standard trace of matrices.

As an exercise the reader should show that Tr is a chain map. Let $i\colon A \to M_k(A)$ be the map that sends a in A to the matrix with only one nonzero component in the upper left corner equal to a. There is a similar map $M \to M_k(M)$. These induce a map

$$I\colon C_n(A, M) \to C_n(M_k(A), M_k(M)),$$

$$I(m \otimes a_1 \otimes \cdots \otimes a_n) = i(m) \otimes i(a_1) \otimes \cdots \otimes i(a_n).$$

We have $\mathrm{Tr} \circ I = \mathrm{id}$, which is easily checked. It is however *not* true that $I \circ \mathrm{Tr} = \mathrm{id}$. There is instead a homotopy between $I \circ \mathrm{Tr}$ and id (cf. [56]). It follows that Tr and I induce inverse isomorphisms between homologies.

As a special case of Morita invariance, by choosing $M = A$, we obtain an isomorphism of Hochschild homology groups

$$HH_n(A) \simeq HH_n(M_k(A))$$

for all n and k.

Example 0.12.6 (Inner derivations and inner automorphisms)**.** We need to know, for example when defining the noncommutative Chern character later in this chapter, that *inner automorphisms* act by the identity on Hochschild homology and *inner derivations* act by zero. Let A be an algebra, $u \in A$ be an invertible element and let $a \in A$ be any element. They induce the chain maps $\Theta\colon C_n(A) \to C_n(A)$ and $L_a\colon C_n(A) \to C_n(A)$ defined by

$$\Theta(a_0 \otimes \cdots \otimes a_n) = ua_0u^{-1} \otimes \cdots \otimes ua_nu^{-1},$$

and

$$L_a(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} a_0 \otimes \cdots \otimes [a, a_i] \otimes \cdots \otimes a_n.$$

Lemma 0.12.1. *Θ induces the identity map on Hochschild homology and L_a induces the zero map.*

Proof. The maps $h_i\colon A^{\otimes n+1} \to A^{\otimes n+2}$, $i = 0, \ldots, n$,

$$h_i(a_0 \otimes \cdots \otimes a_n) = (a_0u^{-1} \otimes ua_1u^{-1} \otimes \cdots \otimes u \otimes a_{i+1} \otimes \cdots \otimes a_n)$$

define a homotopy

$$h = \sum_{i=0}^{n} (-1)^i h_i$$

between id and Θ.

For the second part one checks that the maps $h'_i\colon A^{\otimes n+1} \to A^{\otimes n+2}$, $i = 0, \ldots, n$,

$$h_i(a_0 \otimes \cdots \otimes a_n) = (a_0 \otimes \cdots \otimes a_i \otimes a \otimes \cdots \otimes a_n),$$

define a homotopy

$$h' = \sum_{i=0}^{n} (-1)^i h'_i$$

between L_a and 0. ∎

Exercise 0.12.1. Let $A = S(V)$ be the symmetric algebra of a vector space V. Show that its module of Kähler differentials $\Omega^1_S(V)$ is isomorphic to $S(V) \otimes V$, the free left $S(V)$-module generated by V, where the universal differential is given by

$$d(v_1 v_2 \dots v_n) = \sum_{i=1}^{n} (v_1 \dots \hat{v}_i \dots v_n) \otimes v_i.$$

Exercise 0.12.2 (Additivity of HH_*)**.** Show that for unital algebras A and B, there is a natural isomorphism

$$HH_n(A \oplus B) \simeq HH_n(A) \oplus HH_n(B)$$

for all $n \geq 0$.

Exercise 0.12.3. Show that non-inner automorphisms need not act by the identity on HH_*.

Lecture 3: Cyclic cohomology

0.13 Cyclic Cohomology

To define a noncommutative de Rham type theory for noncommutative algebras is a highly non-trivial matter. Note that the usual algebraic formulation of de Rham theory is based on the module of Kähler differentials and its exterior algebra, which has no analogue for noncommutative algebras. This is in sharp contrast with the situation in K-theory where extending the topological K-theory to noncommutative Banach algebras is straightforward.

Instead, the noncommutative analogue of de Rham homology, called cyclic cohomology by Connes, was found by a careful analysis of the algebraic structures deeply hidden in (*super*)*traces of products of commutators*. These expressions are directly defined in terms of an elliptic operator and its parametrix and were shown, via an index formula, to give the index of the operator when paired with a K-theory class. This connection with elliptic theory, K-homology, and K-theory, will be explored later in these lectures.

Cyclic cohomology is defined in [19] through a remarkable subcomplex of the Hochschild complex. We recall this definition in this section. Later in this chapter we give two other definitions. While these three definitions are equivalent to each other, as we shall see each has its own merits and strengths.

Let A be an algebra over the complex numbers and $(C^*(A), b)$ denote the Hochschild complex of A with coefficients in the A-bimodule A^*. We have, from Section 3.1,

$$C^n(A) = \mathrm{Hom}(A^{\otimes(n+1)}, \mathbb{C}), \quad n = 0, 1, \ldots,$$

and

$$\begin{aligned}(bf)(a_0, \ldots, a_{n+1}) = &\sum_{i=0}^{n} (-1)^i f(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1} f(a_{n+1} a_0, \ldots, a_n)\end{aligned}$$

for all $f \in C^n(A)$.

The following definition is fundamental and marks our departure from Hochschild cohomology:

Definition 0.13.1. An n-cochain $f \in C^n(A)$ is called *cyclic* if

$$f(a_n, a_0, \dots, a_{n-1}) = (-1)^n f(a_0, a_1, \dots, a_n)$$

for all $a_0, \dots, a_n$ in A. We denote the space of cyclic n-cochains on A by $C^n_\lambda(A)$.

Lemma 0.13.1. *The space of cyclic cochains is invariant under the action of b, i.e., for all n we have*

$$b\, C^n_\lambda(A) \subset C^{n+1}_\lambda(A).$$

Proof. Define the operators $\lambda \colon C^n(A) \to C^n(A)$ and $b' \colon C^n(A) \to C^{n+1}(A)$ by

$$(\lambda f)(a_0, \dots, a_n) = (-1)^n f(a_n, a_0, \dots, a_{n-1}),$$
$$(b' f)(a_0, \dots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}).$$

One checks that

$$(1 - \lambda) b = b'(1 - \lambda).$$

Since

$$C^n_\lambda(A) = \operatorname{Ker}(1 - \lambda),$$

the lemma is proved. □

We therefore have a subcomplex of the Hochschild complex, called the *cyclic complex* of A:

$$\boxed{C^0_\lambda(A) \xrightarrow{b} C^1_\lambda(A) \xrightarrow{b} C^2_\lambda(A) \xrightarrow{b} \cdots} \tag{18}$$

Definition 0.13.2. The cohomology of the cyclic complex is called the *cyclic cohomology* of A and will be denoted by $HC^n(A)$, $n = 0, 1, 2, \dots$.

A cocycle for the cyclic cohomology group $HC^n(A)$ is called a *cyclic n-cocycle* on A. It is an $(n+1)$-linear functional f on A which satisfies the two conditions

$$(1 - \lambda) f = 0 \quad \text{and} \quad bf = 0.$$

The inclusion of complexes

$$(C^*_\lambda(A),\, b) \hookrightarrow (C^*(A),\, b)$$

induces a map I from the cyclic cohomology of A to the Hochschild cohomology of A with coefficients in the A-bimodule A^*:

$$I \colon HC^n(A) \to HH^n(A), \quad n = 0, 1, 2, \dots.$$

We shall see that this map is part of a long exact sequence relating Hochschild and cyclic cohomology. For the moment we mention that I need not be injective or surjective (see example below).

Example 0.13.1. Let $A = \mathbb{C}$, the ground field. We have

$$C_\lambda^{2n}(\mathbb{C}) \simeq \mathbb{C}, \quad C_\lambda^{2n+1}(\mathbb{C}) = 0,$$

so the cyclic complex reduces to

$$0 \to \mathbb{C} \to 0 \to \mathbb{C} \to \cdots .$$

It follows that for all $n \geq 0$,

$$HC^{2n}(\mathbb{C}) = \mathbb{C}, \quad HC^{2n+1}(\mathbb{C}) = 0.$$

Since $HH^n(\mathbb{C}) = 0$ for $n \geq 1$, we conclude that the map I need not be injective and the cyclic complex is not a retraction of the Hochschild complex.

Example 0.13.2. It is clear that, for any algebra A, $HC^0(A) = HH^0(A)$ is the space of traces on A.

Example 0.13.3. Let $A = C^\infty(M)$ be the algebra of smooth complex valued functions on a closed smooth oriented manifold M of dimension n. We check that

$$\varphi(f^0, f^1, \ldots, f^n) := \int_M f^0\, df^1 \ldots df^n,$$

is a cyclic n-cocycle on A. We have already checked the cocycle property of φ, $b\varphi = 0$, in Example 0.8.2. The cyclic property of φ

$$\varphi(f^n, f^0, \ldots, f^{n-1}) = (-1)^n \varphi(f^0, \ldots, f^n)$$

is more interesting and is related to *Stokes' formula*. In fact since

$$\int_M (f^n df^0 \ldots df^{n-1} - (-1)^n f^0 df^1 \ldots df^n) = \int_M d(f^n f^0 df^1 \ldots df^{n-1}),$$

we see that the cyclic property of φ follows from a special case of Stokes' formula:

$$\int_M d\omega = 0,$$

valid for any $(n-1)$-form ω on a closed n-manifold M.

The last example can be generalized in several directions. For example, let V be an m-dimensional closed singular chain (a cycle) on M, e.g. V can be a closed m-dimensional submanifold of M. Then integration on V defines an m-dimensional cyclic cocycle on A:

$$\varphi(f^0, f^1, \ldots, f^m) = \int_V f^0\, df^1 \ldots df^m.$$

We obtain a map

$$H_m(M, \mathbb{C}) \to HC^m(C^\infty(M)), \quad m = 0, 1, \ldots,$$

from the singular homology of M (or its equivalents) to the cyclic cohomology of $C^\infty(M)$.

Let

$$\Omega_p M := \mathrm{Hom}_{\mathrm{cont}}(\Omega^p M, \mathbb{C})$$

denote the continuous dual of the space of p-forms on M. Elements of $\Omega_p M$ are *de Rham p-currents* on M as defined in Section 3.1. A p-current is called *closed* if for any $(p-1)$-form ω we have $\langle C, d\omega\rangle = 0$.

It is easy to check that for any m-current C, closed or not, the cochain

$$\varphi_C(f^0, f^1, \ldots, f^m) := \langle C,\, f^0 df^1 \ldots df^m\rangle,$$

is a Hochschild cocycle on $C^\infty(M)$. Now if C is closed, then one can easily check that φ_C is a cyclic m-cocycle on $C^\infty(M)$. We thus obtain natural maps

$$\Omega_m M \to HH^m(C^\infty(M)) \quad \text{and} \quad Z_m M \to HC^m(C^\infty(M)),$$

where $Z_m(M) \subset \Omega_m M$ is the space of closed m-currents on M.

A noncommutative generalization of this procedure involves the notion of a *cycle on an algebra* due to Connes [19] that we recall now. It gives a geometric and intuitively appealing presentation of cyclic cocycles. It also leads to a definition of cup product in cyclic cohomology and the S operator, as we shall indicate later.

Let $(\Omega,\, d)$ be a differential graded algebra. Thus

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \cdots$$

is a graded algebra and $d\colon \Omega^* \to \Omega^{*+1}$ is a square zero *graded derivation* in the sense that

$$d(\omega_1\omega_2) = d(\omega_1)\omega_2 + (-1)^{\deg(\omega_1)}\omega_1 d(\omega_2) \quad \text{and} \quad d^2 = 0$$

for all homogenous elements ω_1 and ω_2 of Ω.

Definition 0.13.3. A *closed graded trace* of dimension n on a differential graded algebra $(\Omega,\, d)$ is a linear map

$$\int : \Omega^n \to \mathbb{C}$$

such that

$$\int d\omega = 0 \quad \text{and} \quad \int (\omega_1\omega_2 - (-1)^{\deg(\omega_1)\deg(\omega_2)}\omega_2\omega_1) = 0$$

for all ω in Ω^{n-1}, ω_1 in Ω^i, ω_2 in Ω^j and $i + j = n$.

Definition 0.13.4. An *n-cycle* over an algebra A is a triple $(\Omega,\, \int,\, \rho)$ where $\int$ is an n-dimensional closed graded trace on $(\Omega,\, d)$ and $\rho\colon A \to \Omega_0$ is an algebra homomorphism.

Given an n-cycle $(\Omega,\ \int,\ \rho)$ over A its *character* is a cyclic n-cocycle on A defined by

$$\varphi(a^0, a^1, \ldots, a^n) = \int \rho(a^0) d\rho(a^1) \ldots d\rho(a^n). \tag{19}$$

Checking the cyclic cocycle conditions $b\varphi = 0$ and $(1-\lambda)\varphi = 0$ is straightforward but instructive. To simplify the notation we drop the homomorphism ρ and write φ as

$$\varphi(a^0, a^1, \ldots, a^n) = \int a^0 da^1 \ldots da^n.$$

We have, using the Leibniz rule for d and the graded trace property of $\int$,

$$\begin{aligned}
&(b\varphi)(a^0, \ldots, a^{n+1}) \\
&\quad = \sum_{i=0}^{n} (-1)^i \int a^0 da^1 \ldots d(a^i a^{i+1}) \ldots da^{n+1} \\
&\qquad + (-1)^{n+1} \int a^{n+1} a^0 da^1 \ldots da^n \\
&\quad = (-1)^n \int a^0 da^1 \ldots da^n \cdot a^{n+1} + (-1)^{n+1} \int a^{n+1} a^0 da^1 \ldots da^n \\
&\quad = 0.
\end{aligned}$$

Notice that we did not need to use the 'closedness' of $\int$ so far. This will be needed however to check the cyclic property of φ:

$$\begin{aligned}
(1-\lambda)\varphi(a^0, \ldots, a^n) &= \int a^0 da^1 \ldots da^n - (-1)^n \int a^n da^0 \ldots da^n \\
&= (-1)^{n-1} \int d(a^n a^0 da^1 \ldots da^{n-1}) \\
&= 0.
\end{aligned}$$

Conversely, one can show that any cyclic cocycle on A is obtained from a cycle over A via (19). To this end, we introduce the algebra $(\Omega A,\ d)$,

$$\Omega A = \Omega^0 A \oplus \Omega^1 A \oplus \Omega^2 A \oplus \cdots,$$

called the algebra of *noncommutative differential forms* on A as follows. ΩA is the universal (non-unital) differential graded algebra generated by A as a subalgebra. We put $\Omega^0 A = A$, and let $\Omega^n A$ be linearly generated over $\mathbb{C}$ by expressions $a_0 da_1 \ldots da_n$ and $da_1 \ldots da_n$ for $a_i \in A$ (cf. [19] for details). Notice that even if A is unital, ΩA is *not* a unital algebra and in particular the unit of A is only an idempotent in ΩA. The differential d is defined by

$$d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n \quad \text{and} \quad d(da_1 \ldots da_n) = 0.$$

The universal property of $(\Omega A,\, d)$ is the fact that for any (not necessarily unital) differential graded algebra $(\Omega,\, d)$ and any algebra map $\rho\colon A \to \Omega^0$ there is a unique extension of ρ to a morphism of differential graded algebras,

$$\hat{\rho}\colon \Omega A \to \Omega. \tag{20}$$

Now given a cyclic n-cocycle φ on A, define a linear map $\int_\varphi\colon \Omega^n A \to \mathbb{C}$ by

$$\int_\varphi (a_0 + \lambda 1)\, da_1 \dots da_n = \varphi(a_0, \dots, a_n).$$

It is easy to check that $\int_\varphi$ is a closed graded trace on ΩA whose character is φ.

Summarizing, we have shown that the relation

$$\int_\varphi (a_0 + \lambda 1)\, da_1 \dots da_n = \varphi(a_0, a_1, \dots, a_n)$$

defines a one-to-one correspondence:

$$\boxed{\{\text{cyclic } n\text{-cocycles on } A\} \simeq \{\text{closed graded traces on } \Omega^n A\}} \tag{21}$$

Notice that for $n = 0$ we recover the relation in Example 0.13.2 between cyclic 0-cocycles on A and traces on A.

Example 0.13.4 (A 2-cycle on the noncommutative torus)**.** Let $\delta_1,\, \delta_2\colon \mathcal{A}_\theta \to \mathcal{A}_\theta$ denote the canonical derivations of the noncommutative torus and $\tau\colon \mathcal{A}_\theta \to \mathbb{C}$ its canonical trace (cf. Example **??**). It can be shown by a direct computation that the 2-cochain φ defined on $\mathcal{A}_\theta$ by

$$\varphi(a_0, a_1, a_2) = (2\pi i)^{-1} \tau \left(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))\right)$$

is a cyclic 2-cocycle. It can also be realized as the character of the following 2-cycle $(\Omega,\, d, \int)$ on $\mathcal{A}_\theta$ as follows. Let $\Omega = \mathcal{A}_\theta \otimes \bigwedge^* \mathbb{C}^2$ be the tensor product of $\mathcal{A}_\theta$ with the exterior algebra of the vector space $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. The differential d is defined by

$$da = \delta_1(a) e_1 + \delta_2(a) e_2, \quad d(a \otimes e_1) = -\delta_2(a) e_1 \wedge e_2, \quad d(a \otimes e_2) = \delta_1(a) e_1 \wedge e_2.$$

The closed graded trace $\int\colon \Omega^2 \to \mathbb{C}$ is defined by

$$\int a \otimes e_1 \wedge e_2 = (2\pi i)^{-1} \tau(a).$$

The graded trace property of $\int$ is a consequence of the trace property of τ and its closedness follows from the invariance of τ under the infinitesimal automorphisms δ_1 and δ_2, that is, the property $\tau(\delta_i(a)) = 0$ for all a and $i = 1, 2$. Now it is clear that the character of this cycle is the cyclic 2-cocycle φ defined above:

$$\int a_0\, da_1 da_2 = \varphi(a_0, a_1, a_2).$$

In the remainder of this section we indicate a variety of different sources of cyclic cocycles, e.g. from group cocycles or Lie algebra cycles.

Example 0.13.5 (From group cocycles to cyclic cocycles)**.** Let G be a discrete group and $A = \mathbb{C}G$ be its group algebra. Let $c(g_1, \ldots, g_n)$ be a group n-cocycle on G. Thus $c\colon G^n \to \mathbb{C}$ satisfies the cocycle condition

$$g_1 c(g_2, \ldots, g_{n+1}) - c(g_1 g_2, \ldots, g_{n+1}) + \cdots + (-1)^{n+1} c(g_1, \ldots, g_n) = 0$$

for all $g_1, \ldots, g_{n+1}$ in G. Assume c is *normalized* in the sense that

$$c(g_1, \ldots, g_n) = 0$$

if $g_i = e$ for some i, or if $g_1 g_2 \ldots g_n = e$. (It can be shown that any cocycle is cohomologous to a normalized one). One checks that

$$\varphi_c(g_0, \ldots, g_n) = \begin{cases} c(g_1, \ldots, g_n), & \text{if } g_0 g_1 \ldots g_n = e, \\ 0 & \text{otherwise,} \end{cases}$$

is a cyclic n-cocycle on the group algebra $\mathbb{C}G$ (cf. [18], [21], or exercises below). In this way one obtains a map from the group cohomology of G to the cyclic cohomology of $\mathbb{C}G$,

$$H^n(G, \mathbb{C}) \to HC^n(\mathbb{C}G), \quad c \mapsto \varphi_c.$$

By a theorem of Burghelea [11], the cyclic cohomology group $HC^n(\mathbb{C}G)$ decomposes over the conjugacy classes of G and the component corresponding to the conjugacy class of the identity contains the group cohomology $H^n(G, \mathbb{C})$ as a summand. (See Example 0.16.3 in this chapter.)

Example 0.13.6 (From Lie algebra homology to cyclic cohomology)**.** We start with a simple special case. Let A be an algebra, $\tau\colon A \to \mathbb{C}$ be a trace, and let $\delta\colon A \to A$ be a derivation on A. We assume that the trace is invariant under the action of the derivation in the sense that

$$\tau(\delta(a)) = 0$$

for all $a \in A$. Then one checks that

$$\varphi(a_0, a_1) := \tau(a_0 \delta(a_1))$$

is a cyclic 1-cocycle on A. A simple commutative example of this is when $A = C^\infty(S^1)$, τ corresponds to the Haar measure, and $\delta = \frac{d}{dx}$. Then one obtains the fundamental class of the circle

$$\varphi(f_0, f_1) = \int f_0\, df_1.$$

See below for a noncommutative example, with $A = \mathcal{A}_\theta$, the smooth noncommutative torus.

This construction can be generalized. Let $\delta_1, \dots, \delta_n$ be a *commuting* family of derivations on A, and let τ be a trace on A which is invariant under the action of the δ_i, $i = 1, \dots, n$. Then one can check that

$$\varphi(a_0, \dots, a_n) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \tau(a_0 \delta_{\sigma(1)}(a_1) \dots \delta_{\sigma(n)}(a_n)) \tag{22}$$

is a cyclic n-cocycle on A. Again we give a commutative example and postpone a noncommutative example to below. Let $A = C_c^\infty(\mathbb{R}^n)$ be the algebra of smooth compactly supported functions on $\mathbb{R}^n$. Let $\tau(f) = \int_{\mathbb{R}^n} f$ and $\delta_i = \frac{\partial}{\partial x_i}$. The corresponding cyclic cocycle is given, using the wedge product, by the formula

$$\varphi(f_0, \dots, f_n) = \int_{\mathbb{R}^n} f_0 \, df_1 \wedge df_2 \wedge \dots \wedge df_n,$$

where $df = \sum_i \frac{\partial f}{\partial x_i} \, dx^i$.

Everything we did so far in this example lends itself to a grand generalization as follows. Let $\mathfrak{g}$ be a Lie algebra acting by derivations on an algebra A. This means that we have a Lie algebra map

$$\mathfrak{g} \to \operatorname{Der}(A, A)$$

from $\mathfrak{g}$ to the Lie algebra of derivations of A. Let $\tau \colon A \to \mathbb{C}$ be a trace which is invariant under the action of $\mathfrak{g}$, i.e.,

$$\tau(X(a)) = 0 \quad \text{for all } X \in \mathfrak{g},\ a \in A.$$

For each $n \geq 0$, define a linear map

$$\bigwedge^n \mathfrak{g} \to C^n(A), \quad c \mapsto \varphi_c,$$

where

$$\varphi_c(a_0, a_1, \dots, a_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \tau(a_0 X_{\sigma(1)}(a_1) \dots X_{\sigma(n)}(a_n)) \tag{23}$$

if $c = X_1 \wedge \dots \wedge X_n$ and extended linearly.

It can be shown that φ_c is a Hochschild cocycle for any c, and that it is a cyclic cocycle if c is a Lie algebra cycle. (See Exercise 0.13.5.)

We therefore obtain, for each $n \geq 0$, a map

$$\chi_\tau \colon H_n^{\text{Lie}}(\mathfrak{g}, \mathbb{C}) \to HC^n(A), \quad c \mapsto \varphi_c,$$

from the Lie algebra homology of $\mathfrak{g}$ with trivial coefficients to the cyclic cohomology of A [18].

In particular if $\mathfrak{g}$ is abelian then of course $H_n^{\text{Lie}}(\mathfrak{g}) = \bigwedge^n(\mathfrak{g})$ and we recover our previously defined map (22):

$$\bigwedge^n(\mathfrak{g}) \to HC^n(A), \quad n = 0, 1, \dots.$$

Here is an example of this construction which first appeared in [16]. Let $A = \mathcal{A}_\theta$ denote the "algebra of smooth functions" on the noncommutative torus. Let $X_1 = (1,0), X_2 = (0,1)$. There is an action of the abelian Lie algebra $\mathbb{R}^2$ on $\mathcal{A}_\theta$ defined on generators of $\mathcal{A}_\theta$ by

$$X_1(U) = U, \quad X_1(V) = 0,$$
$$X_2(U) = 0, \quad X_2(V) = V.$$

The induced derivations on $\mathcal{A}_\theta$ are given by

$$X_1\Big(\sum a_{m,n}U^mV^n\Big) = \sum m a_{m,n}U^mV^n,$$
$$X_2\Big(\sum a_{m,n}U^mV^n\Big) = \sum n a_{m,n}U^mV^n.$$

It is easily checked that the trace τ on $\mathcal{A}_\theta$ defined by

$$\tau\Big(\sum a_{m,n}U^mV^n\Big) = a_{0,0}$$

is invariant under the above action of $\mathbb{R}^2$. The generators of $H_*^{\mathrm{Lie}}(\mathbb{R}^2, \mathbb{C})$ are: 1, X_1, X_2, $X_1 \wedge X_2$.

We therefore obtain the following 0-dimensional, 1-dimensional and 2-dimensional cyclic cocycles on $\mathcal{A}_\theta$:

$$\varphi_0(a_0) = \tau(a_0), \quad \varphi_1(a_0,a_1) = \tau(a_0X_1(a_1)), \quad \varphi_1'(a_0,a_1) = \tau(a_0X_2(a_1)),$$
$$\varphi_2(a_0,a_1,a_2) = \tau(a_0(X_1(a_1)X_2(a_2) - X_2(a_1)X_1(a_2))).$$

It is shown in [19] that these classes form a basis for the continuous periodic cyclic cohomology of $\mathcal{A}_\theta$.

Example 0.13.7 (Cup product in cyclic cohomology)**.** As we indicated before, the notion of cycle over an algebra can be used to give a natural definition of a cup product for cyclic cohomology. By specializing one of the variables to the ground field, we obtain the S-operation.

Let $(\Omega, \int, \rho)$ be an m-dimensional cycle on an algebra A and $(\Omega', \int', \rho')$ an n-dimensional cycle on an algebra B. Let $\Omega \otimes \Omega'$ denote the (graded) tensor product of the differential graded algebras Ω and Ω'. By definition, we have

$$(\Omega \otimes \Omega')_k = \bigoplus_{i+j=k} \Omega_i \otimes \Omega'_j,$$

$$d(\omega \otimes \omega') = (d\omega) \otimes \omega' + (-1)^{\deg(\omega)}\omega \otimes (d\omega').$$

Let

$$\int'' \omega \otimes \omega' = \int \omega \int' \omega' \quad \text{if } \deg(\omega) = m, \ \deg(\omega') = n.$$

It is easily checked that $\int''$ is a closed graded trace of dimension $m+n$ on $\Omega \otimes \Omega'$.

Using the universal property (20) of noncommutative differential forms, applied to the map $\rho \otimes \rho' \colon A \otimes B \to \Omega_0 \otimes \Omega_0'$, one obtains a morphism of differential graded algebras

$$(\Omega(A \otimes B),\, d) \to (\Omega \otimes \Omega',\, d).$$

We therefore obtain a closed graded trace of dimension $m+n$ on $(\Omega(A \otimes B),\, d)$. In [19] it is shown that the resulting *cup product* map in cyclic cohomology,

$$\#\colon HC^m(A) \otimes HC^n(B) \to HC^{m+n}(A \otimes B)$$

is well defined.

We give a couple of simple examples of cup product computations.

Example 0.13.8 (The generalized trace map)**.** Let ψ be a trace on B. Then $\varphi \mapsto \varphi \# \psi$ defines a map

$$HC^m(A) \to HC^m(A \otimes B).$$

Explicitly we have

$$(\varphi \# \psi)(a^0 \otimes b^0, \ldots, a^m \otimes b^m) = \varphi(a^0, \ldots, a^m)\psi(b^0 b^1 \ldots b^m).$$

A special case of this construction plays a very important role in cyclic cohomology and noncommutative geometry. Let $\psi = \mathrm{tr}\colon M_n(\mathbb{C}) \to \mathbb{C}$ be the standard trace. Then 'cupping with trace' defines a map

$$HC^m(A) \to HC^m(M_k(A)).$$

Example 0.13.9 (The periodicity operator S)**.** Another important special case of the cup product is when we choose $B = \mathbb{C}$ and ψ to be the fundamental cyclic 2-cocycle on $\mathbb{C}$ defined by

$$\psi(1,1,1) = 1.$$

This leads to an operation of degree 2 on cyclic cohomology:

$$S\colon HC^n(A) \to HC^{n+2}(A), \quad \varphi \mapsto \varphi \# \psi.$$

The formula simplifies to

$$\begin{aligned}(S\varphi)(a^0, \ldots, a^{n+2}) = \int_\varphi a^0 a^1 a^2\, da^3 \ldots da^{n+2} \\ + \int_\varphi a^0 da^1 (a^2 a^3)\, da^4 \ldots da^{n+2} + \cdots \\ + \int_\varphi a^0\, da^1 \ldots da^{i-1} (a^i a^{i+1}) da^{i+2} \ldots da^{n+2} + \cdots \\ + \int_\varphi a^0\, da^1 \ldots da^n (a^{n+1} a^{n+2}).\end{aligned}$$

In the next section we give a different approach to S via the cyclic bicomplex.

So far we have studied the cyclic cohomology of algebras. There is a 'dual' theory called *cyclic homology* which we introduce now. Let A be an algebra and for $n \geq 0$ let
$$C_n(A) = A^{\otimes(n+1)}.$$
For each $n \geq 0$, define the operators
$$b\colon C_n(A) \to C_{n-1}(A), \quad b' : C_n(A) \to C_{n-1}(A), \quad \lambda\colon C_n(A) \to C_n(A)$$
by
$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad + (-1)^n (a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}), \\ b'(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n), \\ \lambda(a_0 \otimes \cdots \otimes a_n) &= (-1)^n (a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

The relation
$$(1-\lambda)b' = b(1-\lambda)$$
can be easily established. Clearly $(C_*(A), b)$ is the Hochschild complex of A with coefficients in the A-bimodule A. Let
$$C_n^\lambda(A) := C_n(A)/\operatorname{Im}(1-\lambda).$$
The relation $(1-\lambda)b' = b(1-\lambda)$ shows that the operator b is well defined on the quotient complex $C_*^\lambda(A)$. The complex
$$(C_*^\lambda(A), b)$$
is called the *cyclic complex* of A and its homology, denoted by $HC_n(A)$, $n = 0, 1, \ldots$, is called the *cyclic homology* of A.

Example 0.13.10. For $n = 0$,
$$HC_0(A) \simeq HH_0(A) \simeq A/[A, A]$$
is the *commutator quotient* of A. Here $[A, A]$ denotes the subspace of A generated by the commutators $ab - ba$, for a and b in A. In particular if A is commutative then $HC_0(A) = A$.

Exercise 0.13.1. Give a description of Hochschild cocycles on A in terms of linear functionals on ΩA similar to (21).

Exercise 0.13.2. Let $\varphi \in HC^0(A)$ be a trace on A. Show that
$$(S\varphi)(a_0, a_1, a_2) = \varphi(a_0 a_1 a_2).$$
Find an explicit formula for $S^n\varphi$ for all n. Let $\varphi \in HC^1(A)$. Express $S\varphi$ in terms of φ.

Exercise 0.13.3 (Area as a cyclic cocycle)**.** Let $f, g\colon S^1 \to \mathbb{R}$ be smooth functions. The map $u \mapsto (f(u), g(u))$ defines a smooth closed curve in the plane. Its signed area is given by $\int f\,dg$. Notice that $\varphi(f, g) = \int f\,dg$ is a cyclic 1-cocycle on $C^\infty(S^1)$.

Exercise 0.13.4. Let $c\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ be the map

$$c((a, b), (c, d)) = ad - bc.$$

Show that c is a normalized group 2-cocycle in the sense of Example 0.13.5. Show that the associated cyclic 2-cocycle on the group algebra $\mathbb{C}\mathbb{Z}^2$ extends to its smooth completion and coincides, up to scale, with the volume form on the two torus.

Exercise 0.13.5. Check that 1) for any c, the cochain φ_c defined in (23) is a Hochschild cocycle, i.e., $b\varphi_c = 0$; 2) if c is a Lie algebra cycle, i.e., if $\delta(c) = 0$, then φ_c is a cyclic cocycle.

0.14 Connes' long exact sequence

Our goal in this section is to establish the long exact sequence of Connes relating Hochschild and cyclic cohomology groups. There is a similar sequence relating Hochschild and cyclic homology. Connes' sequence is the long exact sequence of a short exact sequence and the main difficulty in the proof is to identify the cohomology of the quotient as cyclic cohomology, with a shift in dimension, and to identify the maps.

Let A be an algebra and let C_λ and C denote its cyclic and Hochschild cochain complexes, respectively. Consider the short exact sequence of complexes

$$0 \to C_\lambda \to C \xrightarrow{\pi} C/C_\lambda \to 0. \tag{24}$$

Its associated long exact sequence is

$$\cdots \to HC^n(A) \to HH^n(A) \to H^n(C/C_\lambda) \to HC^{n+1}(A) \to \cdots . \tag{25}$$

We need to identify the cohomology groups $H^n(C/C_\lambda)$. To this end, consider the short exact sequence

$$0 \to C/C_\lambda \xrightarrow{1-\lambda} (C, b') \xrightarrow{N} C_\lambda \to 0, \tag{26}$$

where the operator N is defined by

$$N = 1 + \lambda + \lambda^2 + \cdots + \lambda^n \colon C^n \to C^n.$$

The relations

$$N(1 - \lambda) = (1 - \lambda)N = 0 \quad \text{and} \quad bN = Nb'$$

can be verified and they show that $1-\lambda$ and N are morphisms of complexes in (26). As for the exactness of (26), the only non-trivial part is to show that $\ker(N) \subset \operatorname{im}(1-\lambda)$. But this follows from the relation

$$(1-\lambda)(1+2\lambda+3\lambda^2+\cdots+(n+1)\lambda^n) = N-(n+1)\,\mathrm{id}_{C^n}\,.$$

Now, assuming A is unital, the middle complex (C, b') in (26) can be shown to be exact. In fact we have a contracting homotopy $s\colon C^n \to C^{n-1}$ defined by

$$(s\varphi)(a_0,\ldots,a_{n-1}) = (-1)^{n-1}\varphi(a_0,\ldots,a_{n-1},1),$$

which satisfies

$$b's + sb' = \mathrm{id}\,.$$

The long exact sequence associated to (26) looks like

$$\cdots \to H^n(C/C_\lambda) \to H^n_{b'}(C) \to HC^n(A) \to H^{n+1}(C/C_\lambda) \to H^{n+1}_{b'}(C) \to \cdots\,. \tag{27}$$

Since $H^n_{b'}(C)=0$ for all n, it follows that the connecting homomorphism

$$\delta\colon HC^{n-1}(A) \xrightarrow{\ \sim\ } H^n(C/C_\lambda) \tag{28}$$

is an *isomorphism* for all $n \geq 0$. Using this in (25), we obtain *Connes' long exact sequence* relating Hochschild and cyclic cohomology:

$$\boxed{\cdots \to HC^n(A) \xrightarrow{I} HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} \to HC^{n+1}(A) \to \cdots} \tag{29}$$

The operators B and S can be made more explicit by finding the connecting homomorphisms in the above long exact sequences. Notice that B is the composition of maps from (25) and (28):

$$B : HH^n(A) \xrightarrow{\pi} H^n(C/C_\lambda) \xrightarrow{\delta^{-1}} HC^{n-1}(A).$$

We have, on the level of cohomology, $B=(1-\lambda)^{-1}b'N^{-1}$. Remarkably this can be expressed, on the level of cochains, by *Connes' operator* B:

$$B = Ns(1-\lambda).$$

In fact, we have

$$\begin{aligned} Ns(1-\lambda)(1-\lambda)^{-1}b'N^{-1}[\varphi] &= Nsb'N^{-1}[\varphi] \\ &= N(1-b's)N^{-1}[\varphi] \\ &= (1-bNsN^{-1})[\varphi] \\ &= [\varphi]. \end{aligned}$$

We can also write B as

$$B = Ns(1-\lambda) = NB_0,$$

where $B_0 \colon C^n \to C^{n-1}$ is defined by

$$B_0\varphi(a_0, \ldots, a_{n-1}) = \varphi(1, a_0, \ldots, a_{n-1}) - (-1)^n \varphi(a_0, \ldots, a_{n-1}, 1).$$

Using the relations $(1-\lambda)b = b'(1-\lambda)$, $(1-\lambda)N = N(1-\lambda) = 0$, $bN = Nb'$, and $sb' + b's = 1$, it is easy to show that

$$bB + Bb = 0 \quad \text{and} \quad B^2 = 0.$$

Let

$$S' \colon HC^{n-1}(A) \to HC^{n+1}(A)$$

be the composition of connecting homomorphisms in (28) and (25) associated to the short exact sequences (24) and (26):

$$S' \colon HC^{n-1}(A) \ \xrightarrow{\ \simeq\ }\ H^n(C/C_\lambda) \to HC^{n+1}(A)$$

Therefore we have

$$S'[\varphi] = [b(1-\lambda)^{-1} b' N^{-1} \varphi].$$

Any cochain $\psi \in (1-\lambda)^{-1} b' N^{-1} \varphi$ has the property that $b\psi$ is cyclic, as can be easily checked, and $B[\psi] = [\varphi]$. For the latter notice that

$$\begin{aligned} B(1-\lambda)^{-1} b' N^{-1} \varphi &= Ns(1-\lambda)(1-\lambda)^{-1} b' N^{-1} \varphi \\ &= N(1-b's) N^{-1} \varphi \\ &= \varphi - bNs\varphi. \end{aligned}$$

This gives us the formula

$$S'[\varphi] = [b\psi] = [bB^{-1}\varphi].$$

So far we have a long exact sequence

$$\cdots \to HC^n(A) \xrightarrow{I} HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S'} HC^{n+1}(A) \to \cdots . \tag{30}$$

At this point an important remark is in order. The operator S' as defined above, coincides, *up to scale*, with the periodicity operator S defined in Example 0.13.9. In fact, using the explicit formulae for both S and S', one shows (cf. also [18], Lemma 4.34) that for any cyclic $(n-1)$-cocycle $[\varphi] \in HC^{n-1}(A)$,

$$S[\varphi] = n(n+1) S'[\varphi].$$

Thus in the exact sequence (30) we can replace S' with its scalar multiple S and this of course will give Connes' exact sequence (29). For future use we record the new formula for $S \colon HC^{n-1}(A) \to HC^{n+1}(A)$,

$$S[\varphi] = n(n+1)\, bB^{-1}[\varphi] = n(n+1)[b(1-\lambda)^{-1} b' N^{-1} \varphi]. \tag{31}$$

Using the periodicity operator S, the *periodic cyclic cohomology* of an algebra A is defined as the *direct limit* under the operator S of cyclic cohomology groups:

$$HP^i(A) := \varinjlim HC^{2n+i}(A), \quad i = 0, 1.$$

Notice that since S has degree 2 there are only two periodic groups.

Typical applications of Connes' *IBS* long exact sequence are to extract information on cyclic cohomology from Hochschild cohomology. We list some of them:

1) Let $f\colon A \to B$ be an algebra homomorphism and suppose that the induced maps on Hochschild groups

$$f^*\colon HH^n(B) \to HH^n(A)$$

are isomorphisms for all $n \geq 0$. Then

$$f^*\colon HC^n(B) \to HC^n(A)$$

is an isomorphism for all $n \geq 0$ as well. This simply follows by comparing the *IBS* sequences for A and B and applying the Five Lemma. For example, using Lemma 0.12.1, it follows that inner automorphisms act as identity on (periodic) cyclic cohomology.

Maps between cohomology groups need not be induced by algebra maps. For example if $f\colon (C^*(B), b) \to (C^*(A), b)$ is a morphism of Hochschild complexes and if f commutes with the cyclic operator λ, then it induces a map $(C^*_\lambda(B), b) \to (C^*_\lambda(A), b)$ between cyclic complexes. Using the *IBS* sequence, we conclude that if the induced maps between Hochschild cohomology groups are isomorphisms then the induced maps between cyclic groups are isomorphisms as well. For example using Lemma 0.12.1 we conclude that derivations act trivially on (periodic) cyclic cohomology groups.

2) (Morita invariance of cyclic cohomology) Let A and B be Morita equivalent unital algebras. The Morita invariance property of cyclic cohomology states that there is a natural isomorphism

$$HC^n(A) \simeq HC^n(B), \quad n = 0, 1, \ldots.$$

For a proof of this fact in general see [56]. In the special case where $B = M_k(A)$ a simple proof can be given as follows. Indeed, by Morita invariance of Hochschild cohomology, we know that the inclusion $i\colon A \to M_k(A)$ induces isomorphisms on Hochschild groups and therefore on cyclic groups by 1) above.

3) (Normalization) A cochain $f\colon A^{\otimes(n+1)} \to \mathbb{C}$ is called *normalized* if

$$f(a_0, a_1, \ldots, a_n) = 0$$

whenever $a_i = 1$ for some $i \geq 1$. It is clear that normalized cyclic cochains form a subcomplex $(C^*_{\lambda,\mathrm{norm}}(A), b)$ of the cyclic complex of A. Since the corresponding inclusion for Hochschild complexes is a quasi-isomorphism (Exercise 0.9.3), using the *IBS* sequence we conclude that the inclusion of cyclic complexes is a quasi-isomorphism as well.

Exercise 0.14.1. Show that (26) is exact (the interesting part is to show that $\operatorname{Ker} N \subset \operatorname{Im}(1-\lambda)$).

Exercise 0.14.2. Prove the relations $bB + Bb = 0$ and $B^2 = 0$. (They will be used later, together with $b^2 = 0$, to define the $(b,\, B)$-bicomplex).

Exercise 0.14.3. Let $A = \mathbb{C}$ be the ground field. Compute the operators $B\colon C^n(\mathbb{C}) \to C^{n+1}(\mathbb{C})$ and $S\colon HC^n(\mathbb{C}) \to HC^{n+2}(\mathbb{C})$. Conclude that $HP^{2n}(\mathbb{C}) = \mathbb{C}$ and $HP^{2n+1}(\mathbb{C}) = 0$.

Exercise 0.14.4. Let $A = M_n(\mathbb{C})$. Show that the cochains $\varphi_{2n}\colon A^{\otimes(2n+1)} \to \mathbb{C}$ defined by

$$\varphi_{2n}(a_0, \ldots, a_{2n}) = \operatorname{Tr}(a_0 a_1 \ldots a_{2n})$$

are cyclic cocycles on A. We have $S[\varphi_{2n}] = \lambda_{2n}[\varphi_{2n+2}]$. Compute the constants λ_{2n}.

Exercise 0.14.5. Give examples of algebras whose Hochschild groups are isomorphic in all dimensions but whose cyclic groups are not isomorphic. In other words, an 'accidental' isomorphism of Hochschild groups does not imply cyclic cohomologies are isomorphic (despite the long exact sequence).

0.15 Connes' spectral sequence

The cyclic complex (18) and the long exact sequence (29), useful as they are, are not powerful enough for computations. A much deeper relation between Hochschild and cyclic cohomology groups is encoded in Connes' (b, B)-*bicomplex* and the associated spectral sequence that we shall briefly recall now, following closely the original paper [19].

Let A be a unital algebra. The (b, B)-bicomplex of A, denoted by $\mathcal{B}(A)$, is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
C^2(A) & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \\
C^1(A) & \xrightarrow{B} & C^0(A) & & \\
\uparrow{\scriptstyle b} & & & & \\
C^0(A). & & & &
\end{array}$$

Of the three relations

$$b^2 = 0, \quad bB + Bb = 0, \quad B^2 = 0,$$

only the middle relation is not obvious. But this follows from the relations $b's + sb' = 1$, $(1-\lambda)b = b'(1-\lambda)$ and $Nb' = bN$, already used in the previous section.

The *total complex* of a bicomplex $(C^{*,*}, d^1, d^2)$ is defined as the complex $(\operatorname{Tot} C, d)$, where $(\operatorname{Tot} C)^n = \bigoplus_{p+q=n} C^{p,q}$ and $d = d^1 + d^2$. The following result is fundamental. It shows that the resulting *Connes' spectral sequence* obtained by filtration by rows which has Hochschild cohomology for its E^1 terms, converges to cyclic cohomology.

Theorem 0.15.1 ([19])**.** *The map $\varphi \mapsto (0, \ldots, 0, \varphi)$ is a quasi-isomorphism of complexes*

$$(C^*_\lambda(A),\, b) \to (\operatorname{Tot} \mathcal{B}(A),\, b + B).$$

This is a consequence of the vanishing of the E^2 term of the second spectral sequence (filtration by columns) of $\mathcal{B}(A)$. To prove this consider the short exact sequence of b-complexes

$$0 \to \operatorname{Im} B \to \operatorname{Ker} B \to \operatorname{Ker} B/\operatorname{Im} B \to 0$$

By a hard lemma of Connes ([19], Lemma 41), the induced map

$$H_b(\operatorname{Im} B) \to H_b(\operatorname{Ker} B)$$

is an isomorphism. It follows that $H_b(\operatorname{Ker} B/\operatorname{Im} B)$ vanishes. To take care of the first column one appeals to the fact that

$$\operatorname{Im} B \simeq \operatorname{Ker}(1-\lambda)$$

is the space of cyclic cochains.

We give an alternative proof of Theorem 0.15.1 above. To this end, consider the *cyclic bicomplex* $\mathcal{C}(A)$ defined by

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
C^2(A) & \xrightarrow{1-\lambda} & C^2(A) & \xrightarrow{N} & C^2(A) & \xrightarrow{1-\lambda} & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \\
C^1(A) & \xrightarrow{1-\lambda} & C^1(A) & \xrightarrow{N} & C^1(A)(A) & \xrightarrow{1-\lambda} & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \\
C^0(A) & \xrightarrow{1-\lambda} & C^0(A) & \xrightarrow{N} & C^0(A)(A) & \xrightarrow{1-\lambda} & \cdots.
\end{array}$$

The total cohomology of $\mathcal{C}(A)$ is isomorphic to cyclic cohomology:

$$H^n(\operatorname{Tot} \mathcal{C}(A)) \simeq HC^n(A), \quad n \geq 0.$$

This is a consequence of the fact that the rows of $\mathcal{C}(A)$ are exact except in degree zero. To see this, define the homotopy operator

$$H = \frac{1}{n+1}(1 + 2\lambda + 3\lambda^2 + \cdots + (n+1)\lambda^n)\colon C^n(A) \to C^n(A). \tag{32}$$

We have $(1-\lambda)H = \frac{1}{n+1}N - \mathrm{id}$, which of course implies the exactness of rows in positive degrees and for the first column we are left with the cyclic complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
C^2_\lambda(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\uparrow b & & \uparrow & & \uparrow & & \\
C^1_\lambda(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\uparrow b & & \uparrow & & \uparrow & & \\
C^0_\lambda(A) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots .
\end{array}$$

We are therefore done with the proof of Theorem 0.15.1 provided we can prove that $\operatorname{Tot}\mathcal{B}(A)$ and $\operatorname{Tot}\mathcal{C}(A)$ are quasi-isomorphic. The next proposition proves this by an explicit formula:

Proposition 0.15.1. *The complexes* $\operatorname{Tot}\mathcal{B}(A)$ *and* $\operatorname{Tot}\mathcal{C}(A)$ *are homotopy equivalent.*

Proof. We define explicit chain maps between these complexes and show that they are chain homotopic via explicit homotopies. Define

$$I\colon \operatorname{Tot}\mathcal{B}(A) \to \operatorname{Tot}\mathcal{C}(A), \quad I = \mathrm{id} + Ns,$$
$$J\colon \operatorname{Tot}\mathcal{C}(A) \to \operatorname{Tot}\mathcal{B}(A), \quad J = \mathrm{id} + sN.$$

One checks that I and J are chain maps.

Now consider the operators

$$g\colon \operatorname{Tot}\mathcal{B}(A) \to \operatorname{Tot}\mathcal{B}(A), \quad g = Ns^2B_0,$$
$$h\colon \operatorname{Tot}\mathcal{C}(A) \to \operatorname{Tot}\mathcal{C}(A), \quad h = s,$$

where $B_0 = s(1-\lambda)$.

We have, by direct computation:

$$I \circ J = \mathrm{id} + h\delta + \delta h,$$
$$J \circ I = \mathrm{id} + g\delta' + \delta' g,$$

where δ (resp. δ') denotes the differential of $\operatorname{Tot}\mathcal{C}(A)$ (resp. $\operatorname{Tot}\mathcal{B}(A)$). □

There is a similar result for cyclic homology.

0.16 Cyclic cohomology computations

Cyclic cohomology has been computed for many algebras, most notably algebras of smooth functions, group algebras and crossed product algebras, groupoid algebras, noncommutative tori, universal enveloping algebras, and almost commutative algebras. Equipped with these core examples, one can then use general results like additivity, Morita invariance, homotopy invariance, Künneth formulae, and excision [31], to compute the (periodic) cyclic cohomology of even larger classes of algebras. The main technique to deal with core examples is to find a suitable resolution for the Hochschild complex to compute the Hochschild cohomology first and then find the action of the operator B on the Hochschild complex. In good cases, the E^2-term of the spectral sequence associated with the (b, B)-bicomplex vanishes and one ends up with a computation of cyclic cohomology. To illustrate this idea, we recall some of these computations in this section.

Example 0.16.1 (Algebras of smooth functions)**.** Let $A = C^\infty(M)$ denote the algebra of smooth complex-valued functions on a closed smooth manifold M with its natural Fréchet algebra topology, and let $(\Omega M,\, d)$ denote the de Rham complex of M. Let $C_n(A) = A^{\hat{\otimes}(n+1)}$ denote the space of continuous n-chains on A. We saw in Example 0.12.2 that the map $\mu\colon C_n(A) \to \Omega^n M$ defined by

$$\mu(f_0 \otimes \cdots \otimes f_n) = \frac{1}{n!} f_0 df_1 \wedge \cdots \wedge df_n,$$

induces an isomorphism between the continuous Hochschild homology of A and differential forms on M:

$$HH_n^{\mathrm{cont}}(A) \simeq \Omega^n M.$$

To compute the continuous cyclic homology of A, we first show that under the map μ the operator B corresponds to the de Rham differential d. More precisely, for each integer $n \geq 0$ we have a commutative diagram:

$$\begin{array}{ccc} C_n(A) & \xrightarrow{\mu} & \Omega^n M \\ \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle d} \\ C_{n+1}(A) & \xrightarrow{\mu} & \Omega^{n+1} M. \end{array}$$

We have

$$\begin{aligned} &\mu B(f_0 \otimes \cdots \otimes f_n) \\ &\quad = \mu \sum_{i=0}^{n} (-1)^{ni} (1 \otimes f_i \otimes \cdots \otimes f_{i-1} - (-1)^n f_i \otimes \cdots \otimes f_{i-1} \otimes 1) \\ &\quad = \frac{1}{(n+1)!} \sum_{i=0}^{n} (-1)^{ni} df_i \dots df_{i-1} \\ &\quad = \frac{1}{(n+1)!} (n+1) df_0 \dots df_n = d\mu(f_0 \otimes \cdots \otimes f_n). \end{aligned}$$

It follows that μ defines a morphism of bicomplexes

$$\mathcal{B}(A) \to \Omega(A),$$

where $\Omega(A)$ is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\Omega^2 M & \xleftarrow[d]{} & \Omega^1 M & \xleftarrow[d]{} & \Omega^0 M \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\Omega^1 M & \xleftarrow[d]{} & \Omega^0 M & & \\
\downarrow{\scriptstyle 0} & & & & \\
\Omega^0 M. & & & &
\end{array}$$

Since μ induces isomorphisms on row homologies, it induces isomorphisms on total homologies as well. Thus we have [19]:

$$HC_n^{\text{cont}}(C^\infty(M)) \simeq \Omega^n M / \operatorname{im} d \oplus H_{\text{dR}}^{n-2}(M) \oplus \cdots \oplus H_{\text{dR}}^k(M),$$

where $k = 0$ if n is even and $k = 1$ if n is odd. Notice that the top part, for $n \leq \dim(M)$, consists of the so called *co-closed* differential n-forms on M.

Using the corresponding periodic complexes, one concludes that the continuous periodic cyclic homology of $C^\infty(M)$ is given by

$$HP_k^{\text{cont}}(C^\infty(M)) \simeq \bigoplus_i H_{\text{dR}}^{2i+k}(M), \quad k = 0, 1.$$

There are of course dual results relating continuous cyclic cohomology of $C^\infty(M)$ and de Rham homology of M. Let $(\Omega_* M,\, d)$ denote the complex of de Rham currents on M. Recall from Example 0.12.2 that the map $\Omega_n M \to C^n_{\text{cont}}(A)$ defined by sending a current $C \in \Omega_n M$ to the cochain φ_C, where

$$\varphi_C(f^0, f^1, \ldots, f^n) = \langle C,\, f^0 df^1 \wedge \cdots \wedge df^n \rangle$$

is a quasi-isomorphism. By basically following the same route as above we obtain the following theorem of Connes [19]:

$$\boxed{HC^n_{\text{cont}}(C^\infty(M)) \simeq Z^n(M) \oplus H^{\text{dR}}_{n-2}(M) \oplus \cdots \oplus H^{\text{dR}}_k(M)} \tag{33}$$

where $Z^n(M)$ is the space of closed de Rham n-currents on M and $k = 0$ if n is even and $k = 1$ if n is odd. Finally, for the continuous periodic cyclic cohomology we obtain:

$$\boxed{HP^k_{\text{cont}}(C^\infty(M)) \simeq \bigoplus_i H^{\text{dR}}_{2i+k}(M), \quad k = 0, 1} \tag{34}$$

Now (33) shows that cyclic cohomology is not *homotopy invariant*. In fact while the de Rham cohomology components are homotopy invariant the top component $Z^n(M)$ cannot be homotopy invariant. Formula (34) on the other hand shows that the periodic cyclic cohomology of $C^\infty(M)$ is homotopy invariant. This is a special case of the homotopy invariance of periodic cyclic cohomology [19], [56]. More precisely, for any algebras A and B and *smoothly homotopic* algebra maps $f_0, f_1 \colon A \to B$, we have $f_0^* = f_1^* \colon HP^*(B) \to HP^*(A)$. In particular the algebras A and $A[x]$ have isomorphic periodic cyclic cohomologies.

Example 0.16.2 (Smooth commutative algebras)**.** Let $A = \mathcal{O}(X)$ be the coordinate ring of an affine smooth variety over $\mathbb{C}$ and let (Ω_A^*, d) denote the de Rham complex of A. As we saw in Example 0.12.1, by Hochschild–Kostant–Rosenberg's theorem, the map

$$\mu \colon C_n(A) \to \Omega_A^n$$

induces an isomorphism between Hochschild homology of A and the differential forms on X. By the same method as in the previous example one then arrives at the isomorphisms

$$HC_n(\mathcal{O}(X)) \simeq \Omega_A^n / \operatorname{im} d \oplus H_{\mathrm{dR}}^{n-2}(X) \oplus \cdots \oplus H_{\mathrm{dR}}^k(X),$$

$$HP_k(\mathcal{O}(X)) \simeq \bigoplus_i H^{\mathrm{dR}}_{2i+k}(X), \quad k = 0, 1.$$

Notice that the de Rham cohomology $H_{\mathrm{dR}}^n(X)$ appearing on the right side is isomorphic to the singular cohomology $H^n(X_{top}, \mathbb{C})$ of the underlying topological space of X.

When X is singular the relations between the cyclic homology of $\mathcal{O}(X)$ and the topology of X_{top} can be quite complicated. The situation for periodic cyclic homology however is quite straightforward, as the following theorem of Feigin and Tsygan [36] indicates:

$$HP_k(\mathcal{O}(X)) \simeq \bigoplus_i H^{2i+k}(X_{\mathrm{top}}, \mathbb{C}). \tag{35}$$

Notice that X need not be smooth and the cohomology on the right-hand side is the singular cohomology.

Example 0.16.3 (Group algebras)**.** Let $\mathbb{C}G$ denote the group algebra of a discrete group G. Here, to be concrete, we work over $\mathbb{C}$, but results hold over any field of characteristic zero. As we saw in Example 0.12.3, the Hochschild complex of $\mathbb{C}G$ decomposes over the set $\langle G \rangle$ of conjugacy classes of G and the homology of each summand is isomorphic to the group homology of a group associated to the conjugacy class:

$$HH_n(\mathbb{C}G) \simeq \bigoplus_{\langle G \rangle} H_n(C_g),$$

where C_g is the centralizer of a representative g of a conjugacy class of G [11], [21], [56]. Recall the decomposition

$$C_*(\mathbb{C}G, b) = \bigoplus_{c \in \langle G \rangle} B(G, c)$$

of the Hochschild complex of $\mathbb{C}G$ from Example 0.12.3, where for each conjugacy class $c \in \langle G \rangle$, $B_n(G,c)$ is the linear span of all $(n+1)$-tuples $(g_0, g_1, \ldots, g_n) \in G^{n+1}$ such that

$$g_0 g_1 \ldots g_n \in c.$$

It is clear that $B_n(G,c)$, $n = 0, 1, 2, \ldots$, are invariant not only under the Hochschild differential b, but also under the cyclic operator λ. Let

$$B_n^\lambda(G,c) = B_n(G,c)/\operatorname{im}(1-\lambda).$$

We then have a decomposition of the cyclic complex of $\mathbb{C}G$ into subcomplexes indexed by conjugacy classes:

$$C_*^\lambda(\mathbb{C}G, b) = \bigoplus_{c \in \hat{G}} B_*^\lambda(G,c).$$

The homology of $B_*^\lambda(G,\{e\})$ was first computed by Karoubi [49], [56] in terms of the group homology of G. The result is

$$H_n(B_*^\lambda(G,\{e\}) = \bigoplus_i H_{n-2i}(G).$$

Burghelea's computation of the cyclic homology of $\mathbb{C}G$ [11] (cf. also [**?**], [**?**] for a purely algebraic proof) can be described as follows. Let $\langle G \rangle^{\text{fin}}$ and $\langle G \rangle^{\infty}$ denote the set of conjugacy classes of elements of finite, and infinite orders, respectively. For an element $g \in G$, let $N_g = C_g/\langle g \rangle$, where $\langle g \rangle$ is the group generated by g and C_g is the centralizer of g. Notice that the isomorphism type of N_g only depends on the conjugacy class of g. In each conjugacy class c we pick a representative $g \in c$ once and for all. Now if g is an element of finite order we have

$$H_n(B_*^\lambda(G,c)) = \bigoplus_{i \geq 0} H_{n-2i}(C_g).$$

On the other hand, if g is of infinite order we have

$$H_n(B_*^\lambda(G,c)) = H_n(N_g).$$

Putting these results together we obtain:

$$\boxed{HC_n(\mathbb{C}G) \simeq \bigoplus_{\langle G \rangle^{\text{fin}}} \big(\bigoplus_{i \geq 0} H_{n-2i}(C_g) \big) \bigoplus_{\langle G \rangle^{\infty}} H_n(N_g)} \tag{36}$$

In particular, the Hochschild group has $H_n(G)$ as a direct summand, while the cyclic homology group has $\bigoplus_i H_{n-2i}(G)$ as a direct summand (corresponding to the conjugacy class of the identity of G).

Example 0.16.4 (Noncommutative torus)**.** We shall briefly recall Connes' computation of the Hochschild and cyclic cohomology groups of smooth noncommutative tori [19]. In Example **??** we showed that when θ is rational the smooth noncommutative torus $\mathcal{A}_\theta$ is Morita equivalent to $C^\infty(T^2)$, the algebra of smooth functions on the torus. One can then use the Morita invariance of Hochschild and cyclic cohomology to reduce the computation of these groups to those for the algebra $C^\infty(T^2)$. This takes care of the computation for rational θ. So, through the rest of this example we assume that θ is irrational and we denote the generators of $\mathcal{A}_\theta$ by U_1 and U_2 with the relation $U_2U_1 = \lambda U_1 U_2$, where $\lambda = e^{2\pi i\theta}$.

Let $\mathcal{B} = \mathcal{A}_\theta \hat{\otimes} \mathcal{A}_\theta^{\mathrm{op}}$. There is a topological free resolution of $\mathcal{A}_\theta$ as a left $\mathcal{B}$-module

$$\mathcal{A}_\theta \xleftarrow{\varepsilon} \mathcal{B} \otimes \Omega_0 \xleftarrow{b_1} \mathcal{B} \otimes \Omega_1 \xleftarrow{b_2} \mathcal{B} \otimes \Omega_2 \leftarrow 0,$$

where $\Omega_i = \bigwedge^i \mathbb{C}^2, i = 0, 1, 2$ is the i-th exterior power of $\mathbb{C}^2$. The differentials are given by

$$\begin{aligned} b_1(1 \otimes e_j) &= 1 \otimes U_j - U_j \otimes 1, \quad j = 1, 2, \\ b_2(1 \otimes (e_1 \wedge e_2)) &= (U_2 \otimes 1 - \lambda \otimes U_2) \otimes e_1 - (\lambda U_1 \otimes 1 - 1 \otimes U_1) \otimes e_2, \\ \varepsilon(a \otimes b) &= ab. \end{aligned}$$

The following result completely settles the question of continuous Hochschild cohomology of A_θ when θ is irrational. Recall that an irrational number θ is said to satisfy a Diophantine condition if $|1 - \lambda^n|^{-1} = O(n^k)$ for some positive integer k.

Proposition 0.16.1 ([19]). *Let $\theta \notin \mathbb{Q}$. Then the following holds.*

a) One has $HH^0(\mathcal{A}_\theta) = \mathbb{C}$.

b) If θ satisfies a Diophantine condition then $HH^i(\mathcal{A}_\theta)$ is 2-dimensional for $i = 1$ and is 1-dimensional for $i = 2$.

c) If θ does not satisfy a Diophantine condition, then $HH^i(\mathcal{A}_\theta)$ are infinite dimensional non-Hausdorff spaces for $i = 1, 2$.

Remarkably, for all values of θ, the periodic cyclic cohomology is finite dimensional and is given by

$$HP^0(\mathcal{A}_\theta) = \mathbb{C}^2, \quad HP^1(\mathcal{A}_\theta) = \mathbb{C}^2.$$

An explicit basis for these groups is given by cyclic 1-cocycles

$$\varphi_1(a_0, a_1) = \tau(a_0 \delta_1(a_1)) \quad \text{and} \quad \varphi_1(a_0, a_1) = \tau(a_0 \delta_2(a_1)),$$

and by cyclic 2-cocycles

$$\varphi(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))) \quad \text{and} \quad S\tau,$$

where $\delta_1, \delta_2 \colon \mathcal{A}_\theta \to \mathcal{A}_\theta$ are the canonical derivations defined by

$$\delta_1\Big(\sum a_{mn} U_1^m U_2^n\Big) = \sum m a_{mn} U_1^m U_2^n \quad \text{and} \quad \delta_2(U_1^m U_2^n) = \sum n a_{mn} U_1^m U_2^n,$$

and $\tau\colon \mathcal{A}_\theta \to \mathbb{C}$ is the canonical trace (cf. Example **??**). Note that $S\tau(a_0, a_1, a_2) = \tau(a_0 a_1 a_2)$.

Let $\mathcal{O}(T^2_\theta)$ denote the (dense) subalgebra of $\mathcal{A}_\theta$ generated by U_1 and U_2. In Exercise 0.9.7 we ask the reader to show that the (algebraic) Hochschild groups of $\mathcal{O}(T^2_\theta)$ are finite dimensional for *all* values of θ.

Exercise 0.16.1. Since $\mathbb{C}\mathbb{Z} = \mathbb{C}[z, z^{-1}]$ is both a group algebra and a smooth algebra, we have two descriptions of its Hochschild and cyclic homologies. Compare the two descriptions and show that they are the same.

Exercise 0.16.2. Let G be a finite group. Use Burghelea's theorem in Example 0.16.3 to compute the Hochschild and cyclic homology of $\mathbb{C}G$. Alternatively, one knows that $\mathbb{C}G$ is a direct sum of matrix algebras and one can use the Morita invariance of Hochschild and cyclic theory. Compare the two approaches.

Exercise 0.16.3. Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$ be the infinite dihedral group. Use (36) to compute the cyclic homology of the group algebra $\mathbb{C}D_\infty$.

Exercise 0.16.4. Let $X = \{(x, y) \in \mathbb{C}^2;\ xy = 0\}$, and let $A = \mathcal{O}(X)$ be the coordinate ring of X. Verify (35) for A.

Exercise 0.16.5. Prove directly, without using Proposition 0.16.1, that when θ is irrational $\mathcal{A}_\theta$ has a unique trace and therefore $HH^0(\mathcal{A}_\theta) = \mathbb{C}$. Describe the traces on $\mathcal{A}_\theta$ for rational θ.

Lecture 4: Cyclic modules

The original motivation of [18] was to define cyclic cohomology of algebras as a derived functor. Since the category of algebras and algebra homomorphisms is not even an additive category (for the simple reason that the sum of two algebra homomorphisms is not an algebra homomorphism in general), the standard (abelian) homological algebra is not applicable. Let k be a unital commutative ring. In [18], the category Λ_k of cyclic k-modules appears as an 'abelianization' of the category of k-algebras. Cyclic cohomology is then shown to be the derived functor of the *functor of traces*, as we explain in this section.

Recall that the *simplicial category* Δ is a small category whose objects are the totally ordered sets ([56])

$$[n] = \{0 < 1 < \cdots < n\}, \quad n = 0, 1, 2, \ldots.$$

A morphism $f\colon [n] \to [m]$ of Δ is an order preserving, i.e., monotone non-decreasing, map $f\colon \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$. Of particular interest among the morphisms of Δ are *faces* δ_i and *degeneracies* σ_j,

$$\delta_i\colon [n-1] \to [n], \quad i = 0, 1, \ldots, n,$$
$$\sigma_j\colon [n+1] \to [n], \quad j = 0, 1, \ldots, n.$$

By definition δ_i is the unique injective morphism missing i and σ_j is the unique surjective morphism identifying j with $j+1$. It can be checked that they satisfy the following *simplicial identities*:

$$\delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{if } i < j,$$
$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \quad \text{if } i \leq j,$$
$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j, \\ \mathrm{id}_{[n]} & \text{if } i = j \text{ or } i = j+1, \\ \delta_{i-1} \sigma_j & \text{if } i > j+1. \end{cases}$$

Every morphism of Δ can be uniquely decomposed as a product of faces followed by a product of degeneracies.

The *cyclic category* Λ has the same set of objects as Δ and in fact contains Δ as a subcategory. Morphisms of Λ are generated by simplicial morphisms and new morphisms $\tau_n\colon [n] \to [n], n \geq 0$, defined by $\tau_n(i) = i+1$ for $0 \leq i < n$ and $\tau_n(n) = 0$. We have the following extra relations:

$$\begin{aligned}
\tau_n\delta_i &= \delta_{i-1}\tau_{n-1}, \quad 1 \leq i \leq n,\\
\tau_n\delta_0 &= \delta_n,\\
\tau_n\sigma_i &= \sigma_{i-1}\tau_{n+1}, \quad 1 \leq i \leq n,\\
\tau_n\sigma_0 &= \sigma_n\tau_{n+1}^2,\\
\tau_n^{n+1} &= \mathrm{id}\,.
\end{aligned}$$

It can be shown that the classifying space $B\Lambda$ of the small category Λ is homotopy equivalent to the classifying space $BS^1 = \mathbb{C}P^\infty$ [18].

A *cyclic object* in a category $\mathcal{C}$ is a functor $\Lambda^{\mathrm{op}} \to \mathcal{C}$. A *cocyclic object* in $\mathcal{C}$ is a functor $\Lambda \to \mathcal{C}$. For any commutative unital ring k, we denote the category of cyclic k-modules by Λ_k. A morphism of cyclic k-modules is a natural transformation between the corresponding functors. Equivalently, a morphism $f\colon X \to Y$ consists of a sequence of k-linear maps $f_n\colon X_n \to Y_n$ compatible with the face, degeneracy, and cyclic operators. It is clear that Λ_k is an abelian category. The kernel and cokernel of a morphism f are defined pointwise: $(\mathrm{Ker}\, f)_n = \mathrm{Ker}\, f_n\colon X_n \to Y_n$ and $(\mathrm{Coker}\, f)_n = \mathrm{Coker}\, f_n\colon X_n \to Y_n$. More generally, if $\mathcal{A}$ is any abelian category then the category $\Lambda\mathcal{A}$ of cyclic objects in $\mathcal{A}$ is itself an abelian category.

Let Alg_k denote the category of unital k-algebras and unital algebra homomorphisms. There is a functor

$$\natural\colon\ \mathrm{Alg}_k \to \Lambda_k$$

defined as follows. To an algebra A we associate the cyclic module $A^\natural$ defined by

$$A_n^\natural = A^{\otimes(n+1)}, \quad n \geq 0,$$

with face, degeneracy and cyclic operators given by

$$\begin{aligned}
\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n,\\
\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1},\\
\sigma_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n,\\
\tau_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.
\end{aligned}$$

A unital algebra map $f\colon A \to B$ induces a morphism of cyclic modules $f^\natural\colon A^\natural \to B^\natural$ by $f^\natural(a_0 \otimes \cdots \otimes a_n) = f(a_0) \otimes \cdots \otimes f(a_n)$, and this defines the functor $\natural$.

Example 0.16.5. We have

$$\mathrm{Hom}_{\Lambda_k}(A^\natural,\, k^\natural) \simeq T(A),$$

where $T(A)$ is the space of traces from $A \to k$. To a trace τ we associate the cyclic map $(f_n)_{n\geq 0}$, where

$$f_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \tau(a_0 a_1 \ldots a_n), \quad n \geq 0.$$

It can be easily shown that this defines a one-to-one correspondence.

Now we can state the following fundamental theorem of Connes [18] which greatly extends the above example:

Theorem 0.16.1. *Let k be a field of characteristic zero. For any unital k-algebra A there is a canonical isomorphism*

$$HC^n(A) \simeq \mathrm{Ext}^n_{\Lambda_k}(A^\natural,\, k^\natural) \quad \textit{for all } n \geq 0.$$

Before sketching its proof, we mention that combined with the above example the theorem implies that cyclic cohomology is, in some sense, the non-abelian derived functor of the functor of traces

$$A \rightsquigarrow T(A)$$

from the category of k-algebras to the category of k-modules.

Sketch of proof. The main step in the proof of Theorem 0.16.1 is to find an *injective resolution* of $k^\natural$ in Λ_k. The required injective cyclic modules will be the dual of some projective cyclic modules that we define first. For each integer $m \geq 0$, let us define a cyclic module $\boldsymbol{C}^m$ where

$$(\boldsymbol{C}^m)_n = k\,\mathrm{Hom}_\Lambda([m],[n])$$

is the free k-module generated by the set of all cyclic maps from $[m] \to [n]$. Composition in Λ defines a natural cyclic module structure on each $\boldsymbol{C}^k$. For any cyclic module M we clearly have $\mathrm{Hom}_{\Lambda_k}(\boldsymbol{C}^m, M) = M_m$. This of course implies that each $\boldsymbol{C}^m$ is a projective cyclic module. (Recall that an object P of an abelian category is called projective if the functor $M \mapsto \mathrm{Hom}(P, M)$ is *exact* in the sense that it sends any short exact sequence in the category into a short exact sequence of abelian groups.) The corresponding projective resolution of $k^\natural$ is defined as the total complex of the following double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\boldsymbol{C}^2 & \xrightarrow{1-\lambda} & \boldsymbol{C}^2 & \xrightarrow{\boldsymbol{N}} & \boldsymbol{C}^2 & \xrightarrow{1-\lambda} & \cdots \\
\boldsymbol{b}\uparrow & & \boldsymbol{b}'\uparrow & & \boldsymbol{b}\uparrow & & \\
\boldsymbol{C}^1 & \xrightarrow{1-\lambda} & \boldsymbol{C}^1 & \xrightarrow{\boldsymbol{N}} & \boldsymbol{C}^1 & \xrightarrow{1-\lambda} & \cdots \\
\boldsymbol{b}\uparrow & & \boldsymbol{b}'\uparrow & & \boldsymbol{b}\uparrow & & \\
\boldsymbol{C}^0 & \xrightarrow{1-\lambda} & \boldsymbol{C}^0 & \xrightarrow{\boldsymbol{N}} & \boldsymbol{C}^0 & \xrightarrow{1-\lambda} & \cdots,
\end{array} \tag{37}$$

where the cyclic module maps $\boldsymbol{b}, \boldsymbol{b}', \boldsymbol{\lambda}$ and $\boldsymbol{N}$ are defined by

$$\boldsymbol{b}(f) = \sum_{i=0}^{k} (-1)^i f \circ \delta_i, \quad \boldsymbol{b}'(f) = \sum_{i=0}^{k-1} (-1)^i f \circ \delta_i,$$

$$\boldsymbol{\lambda}(f) = (-1)^k f \circ \tau_k, \quad \text{and} \quad \boldsymbol{N} = 1 + \lambda + \cdots + \lambda^k$$

Now by a direct argument one shows that the row homologies of the above bicomplex (37) are trivial in positive dimensions. Thus to compute its total homology it remains to compute the homology of the complex of complexes:

$$\boldsymbol{C^0}/(1-\lambda) \xleftarrow{\boldsymbol{b}} \boldsymbol{C^1}/(1-\lambda) \xleftarrow{\boldsymbol{b}} \cdots .$$

It can be shown that for each fixed m, the complex of vector spaces

$$\boldsymbol{C}^{\mathbf{0}}_m/(1-\lambda) \xleftarrow{\boldsymbol{b}} \boldsymbol{C}^{\mathbf{1}}_m/(1-\lambda) \xleftarrow{\boldsymbol{b}} \cdots$$

coincides with the complex that computes the simplicial homology of the simplicial set Δ^m. The simplicial set Δ^m is defined by $\Delta^m_n = \mathrm{Hom}_\Delta([m],[n])$ for all $n \geq 0$. The geometric realization of Δ^m is the closed unit ball in $\mathbb{R}^m$ which is of course contractible. It follows that the total homology of the above bicomplex is the cyclic module $k^\natural$. We note that for this argument K need not be a field. Now if k is a field of characteristic zero the cyclic modules $\boldsymbol{C}_m$, $m \geq 0$, defined by $(\boldsymbol{C}_m)_n = \mathrm{Hom}_k((\boldsymbol{C}^m)_n, k)$ are injective cyclic modules. Dualizing the bicomplex (37), finally we obtain an injective resolution of $k^\natural$ as a cyclic module. To compute the $\mathrm{Ext}^*_{\Lambda_k}(A^\natural, k^\natural)$ groups, we apply the functor $\mathrm{Hom}_{\Lambda_k}(A^\natural, -)$ to this resolution. We obtain the bicomplex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
C^2(A) & \xrightarrow{1-\lambda} & C^2(A) & \xrightarrow{N} & C^2(A) & \xrightarrow{1-\lambda} & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \\
C^1(A) & \xrightarrow{1-\lambda} & C^1(A) & \xrightarrow{N} & C^1 & \xrightarrow{1-\lambda} & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \\
C^0(A) & \xrightarrow{1-\lambda} & C^0(A) & \xrightarrow{N} & C^0(A) & \xrightarrow{1-\lambda} & \cdots .
\end{array} \tag{38}$$

We are done with the proof of Theorem 0.16.1, provided we can show that the cohomology of (38) is isomorphic to the cyclic cohomology of A. But this we have already shown in the last section. This finishes the proof of the theorem. □

A remarkable property of the cyclic category Λ, not shared by the simplicial category, is its *self-duality* in the sense that there is a natural isomorphism of categories $\Lambda \simeq \Lambda^{\mathrm{op}}$ [18]. Roughly speaking, the duality functor $\Lambda^{\mathrm{op}} \to \Lambda$ acts as identity on objects of Λ and exchanges face and degeneracy operators while sending the cyclic operator to its inverse. Thus to a cyclic (resp. cocyclic) module one can associate a cocyclic (resp. cyclic) module by applying the duality isomorphism.

Example 0.16.6 (Hopf cyclic cohomology)**.** We give a very non-trivial example of a cocyclic module. Let H be a Hopf algebra. A character of H is a unital algebra map $\delta \colon H \to \mathbb{C}$. A group-like element is a nonzero element $\sigma \in H$ such that $\Delta\sigma = \sigma \otimes \sigma$. Following [26], [27], we say (δ, σ) is a *modular pair* if $\delta(\sigma) = 1$, and a *modular pair in involution* if

$$\tilde{S}_\delta^2(h) = \sigma h \sigma^{-1}$$

for all h in H. Here the δ-*twisted antipode* $\tilde{S}_\delta \colon H \to H$ is defined by

$$\tilde{S}_\delta(h) = \sum \delta(h^{(1)}) S(h^{(2)}).$$

Now let (H, δ, σ) be a Hopf algebra endowed with a modular pair in involution. In [26] Connes and Moscovici attach a cocyclic module $H^\natural_{(\delta,\sigma)}$ to this data as follows. Let

$$H^{\natural,0}_{(\delta,\sigma)} = \mathbb{C} \quad \text{and} \quad H^{\natural,n}_{(\delta,\sigma)} = H^{\otimes n} \qquad \text{for } n \geq 1.$$

Its face, degeneracy and cyclic operators δ_i, σ_i, and τ_n are defined by

$$\begin{aligned} \delta_0(h_1 \otimes \cdots \otimes h_n) &= 1 \otimes h_1 \otimes \cdots \otimes h_n, \\ \delta_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_n \quad \text{for } 1 \leq i \leq n, \end{aligned}$$

$$\begin{aligned}
\delta_{n+1}(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_n \otimes \sigma, \\
\sigma_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_i \varepsilon(h_{i+1}) \otimes \cdots \otimes h_n \quad \text{for } 0 \leq i < n, \\
\tau_n(h_1 \otimes \cdots \otimes h_n) &= \Delta^{n-1} \tilde{S}_\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma).
\end{aligned}$$

Checking the cyclic property of τ_n, i.e., $\tau_n^{n+1} = 1$ is a highly non-trivial task. The cyclic cohomology of the cocyclic module $H^\natural_{(\delta,\sigma)}$ is the Hopf cyclic cohomology of the triple (H, δ, σ). (cf. also [1], [42], [43] for more examples of cyclic modules arising from actions and coactions of Hopf algebras on algebras and coalgebras.)

Lecture 5: Connes–Chern character

Classicaly, Chern character relates the K-theory of a space to its ordinary cohomology theory. In noncommutative geometry, in addition to K-theory there is also a very important dual *K-homology* theory built out of *abstract elliptic operators* on the noncommutative space. In this lecture we look at the noncommutative analogues of Chern character maps for both K-theory and K-homology, with values in cyclic homology and cyclic cohomology, respectively. As we mentioned before, it was the search for a noncommutative analogue of the Chern character in K-homology that eventually led Alain Connes to the discovery of cyclic cohomology. K-theory, K-homology, cyclic homology and cohomology, via their allied Chern character maps, enter into a beautiful index formula of Connes which plays an important role in applications of noncommutative geometry.

We mentione that of all characteristic classes, the Chern character is the only one that admits an extension to the noncommutative world. In particular there are no analogues of Chern classes or Pontryagin classes for noncommutative algebras.

0.17 What is the Chern character?

The *classical Chern character* is a natural transformation from K-theory to ordinary cohomology theory with rational coefficients [61]. More precisely, for each compact Hausdorff space X we have a natural ring homomorphism

$$\mathrm{Ch}\colon K^0(X) \to \bigoplus_{i\geq 0} H^{2i}(X, \mathbb{Q}),$$

where K^0 (resp. H) denotes the K-theory (resp. Čech cohomology with rational coefficients). Moreover, thanks to result of Atiyah and Hirzebruch, it is known that *Ch is a rational isomorphism*. It satisfies certain axioms and these axioms completely characterize Ch. We shall not recall these axioms here since they are not very useful for finding the noncommutative analogue of Ch. What turned out to be most useful in this regard was the *Chern–Weil* definition of the Chern character for smooth manifolds,

$$\mathrm{Ch}\colon K^0(X) \to \bigoplus_{i\geq 0} H^{2i}_{\mathrm{dR}}(X),$$

using differential geometric notions of connection and curvature on vector bundles over smooth manifolds. (cf. [61], and Example 0.18.4 in this section). Now let us describe the situation in the noncommutative case.

In [16], [19], [21], Connes shows that Chern–Weil theory admits a vast generalization to a noncommutative setting. For example, for a not necessarily commutative algebra A and each integer $n \geq 0$ there are natural maps, called *Connes–Chern character* maps,

$$\mathrm{Ch}^{2n}_0\colon K_0(A) \to HC_{2n}(A),$$

$$\mathrm{Ch}^{2n+1}_1\colon K_1(A) \to HC_{2n+1}(A)$$

from the K-theory of A to its cyclic homology.

Alternatively, these maps can be defined as a *pairing* between cyclic cohomology and K-theory:

$$HC^{2n}(A) \otimes K_0(A) \to \mathbb{C}, \quad HC^{2n+1}(A) \otimes K_1(A) \to \mathbb{C}. \tag{39}$$

These pairings are compatible with the periodicity operator S in cyclic cohomology in the sense that

$$\langle [\varphi],\, [e]\rangle = \langle S[\varphi], [e]\rangle$$

for all cyclic cocycles φ and K-theory classes $[e]$, and thus induce a pairing

$$HP^i(A) \otimes K_i(A) \to \mathbb{C}, \quad i = 0, 1,$$

between periodic cyclic cohomology and K-theory.

0.18 Connes–Chern character in K-theory

We start by defining the pairings (39). Let φ be a *cyclic* $2n$*-cocycle* on an algebra A. For each integer $k \geq 1$, the formula

$$\tilde{\varphi}(m_0 \otimes a_0, \ldots, m_{2n} \otimes a_{2n}) = \mathrm{tr}(m_0 \ldots m_{2n})\varphi(a_0, \ldots, a_{2n}) \tag{40}$$

defines a cyclic $2n$-cocycle $\tilde{\varphi} \in Z^{2n}_\lambda(M_k(A))$. Let $e \in M_k(A)$ be an idempotent representing a class in $K_0(A)$. Define a bilinear map

$$HC^{2n}(A) \otimes K_0(A) \to \mathbb{C} \tag{41}$$

by the following formula:

$$\boxed{\langle [\varphi],\, [e]\rangle = (n!)^{-1}\, \tilde{\varphi}(e, \ldots, e)} \tag{42}$$

Let us first check that the value of the pairing depends only on the cyclic cohomology class of φ in $HC^{2n}(A)$. It suffices to assume $k=1$ (why?). Let $\varphi = b\psi$ with $\psi \in C_\lambda^{2n-1}(A)$, be a coboundary. Then we have

$$\begin{aligned}\varphi(e,\dots,e) &= b\psi(e,\dots,e)\\ &= \psi(ee,e,\dots,e) - \psi(e,ee,\dots,e) + \cdots + (-1)^{2n}\psi(ee,e,\dots,e)\\ &= \psi(e,\dots,e)\\ &= 0,\end{aligned}$$

where the last relation follows from the cyclic property of ψ.

To verify that the value of $\langle[\varphi],[e]\rangle$, for fixed φ, only depends on the class of $[e] \in K_0(A)$ we have to check that for $u \in \mathrm{GL}_k(A)$ an invertible matrix, we have $\langle[\varphi],[e]\rangle = \langle[\varphi],[ueu^{-1}]\rangle$. It again suffices to show this for $k=1$. But this is exactly the fact, proved in Section 3.7, that inner automorphisms act by the identity on cyclic cohomology. Formula (42) can be easily seen to be additive in $[e]$ under the direct sum $e \oplus f$ of idempotents. This shows that the pairing (41) is well defined.

Proposition 0.18.1. *For any cyclic cocycle $\varphi \in Z_\lambda^{2n}(A)$ and idempotent $e \in M_k(A)$ we have*

$$\langle[\varphi],[e]\rangle = \langle S[\varphi],[e]\rangle.$$

Proof. Without loss of generality we can assume that $k=1$. Using our explicit formula (31) for the S-operator, we have

$$S[\varphi] = (2n+1)(2n+2)[b(1-\lambda)^{-1}b'N^{-1}\varphi],$$

where $N^{-1}\varphi = \frac{1}{2n+1}\varphi$ (since φ is cyclic), and

$$(1-\lambda)^{-1}b'\varphi = \frac{-1}{2n+2}(1+2\lambda+3\lambda^2+\cdots+(2n+2)\lambda^{2n+1})b'\varphi.$$

Thus we have

$$\begin{aligned}S\varphi(e,\dots,e) &= -b(1+2\lambda+3\lambda^2+\cdots+(2n+2)\lambda^{2n+1})b'\varphi(e,\dots,e)\\ &= (n+1)b'\varphi(e,\dots,e)\\ &= (n+1)\varphi(e,\dots,e).\end{aligned}$$

Now we have

$$\langle S[\varphi],[e]\rangle = \frac{1}{(n+1)!}(S\varphi)(e,\dots,e) = \frac{1}{n!}\varphi(e,\dots,e) = \langle[\varphi],[e]\rangle. \qquad \square$$

Example 0.18.1 ($n=0$)**.** $HC^0(A)$ is the space of traces on A. Therefore the Connes–Chern pairing for $n=0$ reduces to a map

$$\{\text{traces on } A\} \times K_0(A) \to \mathbb{C},$$

$$\langle \tau, [e] \rangle = \sum_{i=1}^{k} \tau(e_{ii}),$$

where $e = [e_{ij}] \in M_k(A)$ is an idempotent. The induced function on $K_0(A)$ is called the *dimension function* and denoted by $\dim_\tau$. This terminology is suggested by the commutative case. In fact if X is a compact connected topological space, then $\tau(f) = f(x_0)$, $x_0 \in X$, defines a trace on $C(X)$, and for a vector bundle E on X, $\dim_\tau(E)$ is the rank of the vector bundle E and is an integer. One of the striking features of noncommutative geometry is the existence of noncommutative vector bundles with non-integral dimensions. A beautiful example of this phenomenon is shown is Example **??** through the Powers–Rieffel projection $e \in \mathcal{A}_\theta$ with $\tau(e) = \theta$, where τ is the canonical trace on the noncommutative torus (cf. also [21]).

Here is a slightly different approach to this dimension function. Let E be a finite projective right A-module. A trace τ on A induces a trace on the endomorphism algebra of E,

$$\mathrm{Tr}\colon \mathrm{End}_A(E) \to \mathbb{C}$$

as follows. First assume that $E = A^n$ is a free module. Then $\mathrm{End}_A(E) \simeq M_n(A)$ and our trace map is defined by

$$\mathrm{Tr}(a_{i,j}) = \sum_i a_{ii}.$$

It is easy to check that the above map is a trace. In general, there is an A-module F such that $E \oplus F \simeq A^n$ is a free module and $\mathrm{End}_A(E)$ embeds in $M_n(A)$. One can check that the induced trace on $\mathrm{End}_A(E)$ is independent of the choice of splitting. Now, from our description of Tr in terms of τ, it is clear that

$$\langle \tau, [E] \rangle = \dim_\tau(E) = \mathrm{Tr}(\mathrm{id}_E)$$

for any finite projective A-module E.

The topological information hidden in an idempotent is much more subtle than just its 'rank', as two idempotents, say vector bundles, can have the same rank but still be non-isomorphic. In fact traces can only capture the 0-dimensional information. To know more about idempotents and K-theory we need the higher dimensional analogues of traces, which are cyclic cocycles, and the pairing (42).

As we saw in Section 0.15 cyclic cocycles can also be realized in the (b, B)-bicomplex picture of cyclic cohomology. Given an even cocycle

$$\varphi = (\varphi_0, \varphi_2, \ldots, \varphi_{2n})$$

in the (b, B)-bicomplex, its pairing with an idempotent $e \in M_k(\mathcal{A})$ can be shown to be given by

$$\boxed{\langle [\varphi], [e] \rangle = \sum_{k=1}^{n} (-1)^k \frac{k!}{(2k)!} \varphi_{2k}\left(e - \frac{1}{2}, e, \ldots, e\right)} \tag{43}$$

(cf. Exercise 0.18.2).

When A is a Banach (or at least a suitable topological) algebra, to verify that the pairing (42) is well defined, it suffices to check that for a smooth family of idempotents e_t, $0 \leq t \leq 1$, $\varphi(e_t, \ldots, e_t)$ is constant in t. There is an alternative "infinitesimal proof" of this fact which is worth recording [21]:

Lemma 0.18.1. *Let e_t, $0 \leq t \leq 1$, be a smooth family of idempotents in a Banach algebra A. There exists a smooth family x_t, $0 \leq t \leq 1$, of elements of A such that*

$$\dot{e_t} := \frac{d}{dt}(e_t) = [x_t,\, e_t] \quad \textit{for } 0 \leq t \leq 1.$$

Proof. Let

$$x_t = [\dot{e_t},\, e_t] = \dot{e_t}e_t - e_t\dot{e_t}.$$

Differentiating the idempotent condition $e_t^2 = e_t$ with respect to t we obtain

$$\frac{d}{dt}(e_t^2) = \dot{e_t}e_t + e_t\dot{e_t} = \dot{e_t}.$$

Multiplying this last relation on the left by e_t yields

$$e_t\dot{e_t}e_t = 0.$$

Now we have

$$[x_t,\, e_t] = [\dot{e_t}e_t - e_t\dot{e_t},\, e_t] = \dot{e_t}e_t + e_t\dot{e_t} = \dot{e_t}.$$

□

It follows that if $\tau\colon A \to \mathbb{C}$ is a trace (= a cyclic zero cocycle), then

$$\frac{d}{dt}\langle \tau,\, e_t\rangle = \frac{d}{dt}\tau(e_t) = \tau(\dot{e_t}) = \tau([x_t,\, e_t]) = 0.$$

Hence the value of the pairing, for a fixed τ, depends only on the homotopy class of the idempotent. This shows that the pairing

$$\{\text{traces on } A\} \times K_0(A) \to \mathbb{C}$$

is well defined.

This is generalized in

Lemma 0.18.2. *Let $\varphi(a_0, \ldots, a_{2n})$ be a cyclic $2n$-cocycle on A and let e_t be a smooth family of idempotents in A. Then the number*

$$\langle [\varphi],\, [e_t]\rangle = \varphi(e_t, \ldots, e_t)$$

is constant in t.

Proof. Differentiating with respect to t and using the above lemma, we obtain

$$\begin{aligned}\frac{d}{dt}\varphi(e_t,\ldots,e_t) &= \varphi(\dot{e_t},\ldots,e_t)+\varphi(e_t,\dot{e_t},\ldots,e_t)+\cdots\\ &\qquad\cdots+\varphi(e_t,\ldots,e_t,\dot{e_t})\\ &=\sum_{i=0}^{2n}\varphi(e_t,\ldots,[x_t,e_t],\ldots,e_t)\\ &=L_{x_t}\varphi(e_t,\ldots,e_t).\end{aligned}$$

We saw in Section 0.14 that inner derivations act trivially on Hochschild and cyclic cohomology. This means that for each t there is a cyclic $(2n-1)$-cochain ψ_t such that the *Lie derivative* $L_{x_t}\varphi = b\psi_t$. We then have

$$\frac{d}{dt}\varphi(e_t,\ldots,e_t)=(b\psi_t)(e_t,\ldots,e_t)=0. \qquad \square$$

The formulas in the *odd case* are as follows. Given an invertible matrix $u \in M_k(A)$, representing a class in $K_1^{\text{alg}}(A)$, and an odd cyclic $(2n-1)$-cocycle φ on A, we define

$$\langle[\varphi],\,[u]\rangle := \frac{2^{-(2n+1)}}{(n-\frac{1}{2})\ldots\frac{1}{2}}\,\tilde{\varphi}(u^{-1}-1,u-1,\ldots,u^{-1}-1,u-1), \tag{44}$$

where the cyclic cocycle $\tilde{\varphi}$ is defined in (40). As we saw in Section 0.14, any cyclic cocycle can be represented by a *normalized* cocycle for which $\varphi(a_0,\ldots,a_1)=0$ if $a_i = 1$ for some i. When φ is normalized, formula (44) reduces to

$$\boxed{\langle[\varphi],\,[u]\rangle = \frac{2^{-(2n+1)}}{(n-\frac{1}{2})\ldots\frac{1}{2}}\,\tilde{\varphi}(u^{-1},u,\ldots,u^{-1},u)} \tag{45}$$

As in the even case, the induced pairing $HC^{2n+1}(A)\otimes K_1^{\text{alg}}(A)\to\mathbb{C}$ is compatible with the periodicity operator: for any odd cyclic cocycle $\varphi\in Z_\lambda^{2n+1}(A)$ and an invertible $u\in \mathrm{GL}_k(A)$, we have

$$\langle[\varphi],\,[u]\rangle = \langle S[\varphi],[u]\rangle.$$

These pairings are just manifestations of perhaps more fundamental maps that define the even and odd *Connes–Chern characters*

$$\begin{aligned}\mathrm{Ch}_0^{2n}&\colon K_0(A)\to HC_{2n}(A),\\ \mathrm{Ch}_1^{2n+1}&\colon K_1(A)\to HC_{2n+1}(A),\end{aligned}$$

as we describe them now. In the even case, given an idempotent $e=(e_{ij})\in M_k(A)$, we define for each $n\geq 0$,

$$\begin{aligned}\mathrm{Ch}_0^{2n}(e) &= (n!)^{-1}\,\mathrm{Tr}(\underbrace{e\otimes e\otimes\cdots\otimes e}_{2n+1})\\ &=\sum_{i_0,i_1,\ldots,i_{2n}} e_{i_0i_1}\otimes e_{i_1i_2}\otimes\cdots\otimes e_{i_{2n}i_0},\end{aligned} \tag{46}$$

where on the right-hand side the class of the tensor in $A^{\otimes(2n+1)}/\operatorname{Im}(1-\lambda)$ is understood. In low dimensions we have

$$\mathrm{Ch}_0^0(e) = \sum_{i=1}^{k} e_{ii},$$
$$\mathrm{Ch}_0^2(e) = \sum_{i_0=1}^{k} \sum_{i_1=1}^{k} \sum_{i_2=1}^{k} e_{i_0 i_1} \otimes e_{i_1 i_2} \otimes e_{i_2 i_0},$$

etc. To check that $\mathrm{Ch}_0^{2n}(e)$ is actually a cycle, notice that

$$b(\mathrm{Ch}_0^{2n}(e)) = \frac{1}{2}(1-\lambda)\operatorname{Tr}(\underbrace{e \otimes \cdots \otimes e}_{2n}),$$

which shows its class is zero in the quotient.

In the odd case, given an invertible matrix $u \in M_k(A)$, we define

$$\mathrm{Ch}_1^{2n+1}([u]) = \operatorname{Tr}(\underbrace{(u^{-1}-1) \otimes (u-1) \otimes \cdots \otimes (u^{-1}-1) \otimes (u-1)}_{2n+2}).$$

Example 0.18.2. Let $A = C^\infty(S^1)$ denote the algebra of smooth complex-valued functions on the circle. One knows that $K_1(A) \simeq K^1(S^1) \simeq \mathbb{Z}$ and $u(z) = z$ is a generator of this group. Let

$$\varphi(f_0, f_1) = \int_{S^1} f_0 \, df_1$$

denote the cyclic cocycle on A representing the fundamental class of S^1 in de Rham homology. Notice that this is a normalized cocycle since $\varphi(1, f) = \varphi(f, 1) = 0$ for all $f \in A$. We have

$$\langle [\varphi], [u] \rangle = \varphi(u, u^{-1}) = \int_{S^1} u \, du^{-1} = -2\pi i.$$

Alternatively, the Connes–Chern character

$$\mathrm{Ch}_1^1([u]) = u \otimes u^{-1} \in HC_1(A) \simeq H^1_{\mathrm{dR}}(S^1)$$

is the class of the differential form $\omega = z^{-1}dz$, representing the fundamental class of S^1 in de Rham cohomology.

Example 0.18.3. Let $A = C^\infty(S^2)$ and let $e \in M_2(A)$ denote the idempotent representing the Hopf line bundle on S^2:

$$e = \frac{1}{2}\begin{pmatrix} 1+x_3 & x_1+ix_2 \\ x_1-ix_2 & 1-x_3 \end{pmatrix}.$$

Let us check that under the map

$$HC_2(A) \to \Omega^2 S^2, \quad a_0 \otimes a_1 \otimes a_2 \mapsto a_0 da_1 da_2,$$

the Connes–Chern character of e corresponds to the fundamental class of S^2. We have

$$\begin{aligned}
\mathrm{Ch}_0^2(e) &= \mathrm{Tr}(e \otimes e \otimes e) \mapsto \mathrm{Tr}(edede) \\
&= \frac{1}{8}\,\mathrm{Tr}\begin{pmatrix} 1+x_3 & x_1+ix_2 \\ x_1-ix_2 & 1-x_3 \end{pmatrix}\begin{pmatrix} dx_3 & dx_1+idx_2 \\ dx_1-idx_2 & -dx_3 \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} dx_3 & dx_1+idx_2 \\ dx_1-idx_2 & -dx_3 \end{pmatrix}.
\end{aligned}$$

Performing the computation one obtains

$$\mathrm{Ch}_0^2(e) \mapsto \frac{-i}{2}(x_1 dx_2 dx_3 - x_2 dx_1 dx_3 + x_3 dx_1 dx_2).$$

One can then integrate this 2-form on the two-sphere S^2. The result is $-2\pi i$. Notice that for the unit of the algebra $1 \in A$, representing the trivial rank one line bundle on S^2, we have $\mathrm{Ch}_0^0(1) = 1$ and $\mathrm{Ch}_0^{2n}(1) = 0$ for all $n > 0$. Thus e and 1 represent different K-theory classes in $K_0(A)$. A fact which cannot be proved using just $\mathrm{Ch}_0^0(e) = \mathrm{Tr}(e) = 1$.

Example 0.18.4. For smooth *commutative* algebras, the noncommutative Chern character reduces to the classical Chern character. We verify this only in the even case. The verification hinges on two things: the Chern–Weil approach to characteristic classes via connections and curvatures, and the general fact, valid even in the noncommutative case, that an idempotent $e \in M_n(A)$ is more than just a (noncommutative) vector bundle as it carries with it a god-given connection:

$$\boxed{\text{idempotent} = \text{noncommutative vector bundle} + \text{connection}}$$

Let X be a smooth closed manifold, $A = C^\infty(X)$, and let $\Omega^\bullet X$ denote the de Rham complex of X. The alternative definition of the classical Chern character Ch, called the *Chern–Weil theory*, uses the differential geometric notions of connection and curvature on vector bundles as we briefly recall now [61]. Let E be a complex vector bundle on X and let ∇ be a *connection* on E. Thus by definition,

$$\nabla \colon C^\infty(E) \to C^\infty(E) \otimes_A \Omega^1 X$$

is a $\mathbb{C}$-linear map satisfying the Leibniz rule

$$\nabla(f\xi) = f\nabla(\xi) + \xi \otimes df$$

for all smooth sections ξ of E and smooth functions f on X. Let

$$\hat{\nabla} \colon C^\infty(E) \otimes_A \Omega^\bullet X \to C^\infty(E) \otimes_A \Omega^{\bullet+1} X$$

denote the natural extension of ∇ to E-valued differential forms. It is uniquely defined by virtue of the graded Leibniz rule

$$\hat{\nabla}(\xi\omega) = \hat{\nabla}(\xi)\omega + (-1)^{\deg \xi}\xi d\omega$$

for all $\xi \in C^\infty(E) \otimes_A \Omega^\bullet X$ and $\omega \in \Omega^\bullet X$. The *curvature* of ∇ is the operator

$$\hat{\nabla}^2 \in \mathrm{End}_{\Omega^\bullet X}(C^\infty(E) \otimes_A \Omega^\bullet X) = C^\infty(\mathrm{End}(E)) \otimes \Omega^\bullet X,$$

which can be easily checked to be ΩX-linear. Thus it is completely determined by its restriction to $C^\infty(E)$. This gives us the *curvature form* of ∇ as a *'matrix-valued 2-form'*

$$R \in C^\infty(\mathrm{End}(E)) \otimes \Omega^2 X.$$

Let

$$\mathrm{Tr}\colon C^\infty(\mathrm{End}(E)) \otimes_A \Omega^{\mathrm{ev}} X \to \Omega^{\mathrm{ev}} X$$

denote the canonical trace. The Chern character of E is then defined to be the class of the non-homogeneous even form

$$\mathrm{Ch}(E) = \mathrm{Tr}(e^R).$$

(We have omitted the normalization factor of $\frac{1}{2\pi i}$ to be multiplied by R.) One shows that $\mathrm{Ch}(E)$ is a closed form and that its cohomology class is independent of the choice of connection.

Now let $e \in M_n(C^\infty(X))$ be an idempotent representing the smooth vector bundle E on X. Smooth sections of E are in one-to-one correspondence with smooth map $\xi\colon X \to \mathbb{C}^n$ such that $e\xi = \xi$. One can check that the following formula defines a connection on E, called the *Levi-Civita* or *Grassmannian* connection:

$$\nabla(\xi) = ed\xi \in C^\infty(E) \otimes_A \Omega^1 X.$$

Computing the curvature form, we obtain

$$R(\xi) = \hat{\nabla}^2(\xi) = ed(ed\xi) = eded\xi.$$

Differentiating the relation $\xi = e\xi$, we obtain $d\xi = (de)\xi + ed\xi$. Also, by differentiating the relation $e^2 = e$, we obtain $ede \cdot e = 0$. If we use these two relations in the above formula for R, we obtain

$$R(\xi) = edede\xi,$$

and hence the following formula for the matrix valued curvature 2-form R:

$$R = edede.$$

Using $ede \cdot e = 0$, we can easily compute powers of R. They are given by

$$R^n = (edede)^n = e\underbrace{dede\ldots dede}_{2n}.$$

The classical Chern–Weil formula for the Chern character $\mathrm{Ch}(E)$ is

$$\mathrm{Ch}(E) = \mathrm{Tr}(e^R) = \mathrm{Tr}\Big(\sum_{n\geq 0} \frac{R^n}{n!}\Big) \in \Omega^{\mathrm{even}}(X),$$

so that its n-th component is given by

$$\mathrm{Tr}\,\frac{R^n}{n!} = \frac{1}{n!}\,\mathrm{Tr}((ededede)^n) = \frac{1}{n!}\,\mathrm{Tr}(ede\dots de) \in \Omega^{2n}X.$$

The Connes–Chern character of e defined in (46) is

$$\mathrm{Ch}_0^{2n}(e) := (n!)^{-1}\,\mathrm{Tr}(e\otimes\cdots\otimes e).$$

We see that under the canonical map

$$HC_{2n}(A) \to H^{2n}_{\mathrm{dR}}(M), \quad a_0\otimes\cdots\otimes a_{2n} \mapsto a_0da_1\dots da_{2n},$$

$\mathrm{Ch}_0^{2n}(e)$ is mapped to the component of $\mathrm{Ch}(E)$ of degree $2n$.

Example 0.18.5 (Noncommutative Chern–Weil theory)**.** It may happen that an element of $K_0(A)$ is represented by a finite projective module, rather than by an explicit idempotent. It is then important to have a formalism that would give the value of its pairing with cyclic cocycles. This is in fact possible and is based on a noncommutative version of Chern–Weil theory developed by Connes in [16], [19] that we sketch next.

Let A be an algebra. By a *noncommutative differential calculus* on A we mean a triple $(\Omega,\, d, \rho)$ such that $(\Omega,\, d)$ is a differential graded algebra and $\rho\colon A\to\Omega^0$ is an algebra homomorphism. Thus

$$\Omega = \Omega^0\oplus\Omega^1\oplus\Omega^1\oplus\Omega^2\oplus\cdots$$

is a graded algebra, and we assume that the differential $d\colon \Omega^i\to\Omega^{i+1}$ increases the degree, and d is a graded derivation in the sense that

$$d(\omega_1\omega_2) = d(\omega_1)\omega_2 + (-1)^{\deg(\omega_1)}\omega_1 d(\omega_2) \quad \text{and} \quad d^2 = 0.$$

Given a differential calculus on A and a right A-module $\mathcal{E}$, a *connection* on $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{E}\to\mathcal{E}\otimes_A\Omega^1$$

satisfying the Leibniz rule

$$\nabla(\xi a) = \nabla(\xi)a + \xi\otimes da$$

for all $\xi\in\mathcal{E}$ and $a\in A$.

Let

$$\hat{\nabla}\colon \mathcal{E}\otimes_{\mathcal{A}}\Omega^\bullet \to \mathcal{E}\otimes_{\mathcal{A}}\Omega^{\bullet+1}$$

be the (necessarily unique) extension of ∇ which satisfies the graded Leibniz rule

$$\hat{\nabla}(\xi\omega) = \hat{\nabla}(\xi)\omega + (-1)^{\deg \xi}\xi d\omega$$

with respect to the right Ω-module structure on $\mathcal{E} \otimes_{\mathcal{A}} \Omega$. It is defined by

$$\hat{\nabla}(\xi \otimes \omega) = \nabla(\xi)\omega + (-1)^{\deg \omega}\xi \otimes d\omega.$$

The *curvature* of ∇ is the operator $\hat{\nabla}^2 \colon E \otimes_A \Omega^\bullet \to E \otimes_A \Omega^\bullet$, which can be easily checked to be Ω-linear:

$$\hat{\nabla}^2 \in \mathrm{End}_\Omega(E \otimes_A \Omega) = \mathrm{End}_A(E) \otimes \Omega.$$

Let $\int \colon \Omega^{2n} \to \mathbb{C}$ be a closed graded trace representing a cyclic $2n$-cocycle φ on A (cf. Definition 0.13.4). Now since E is finite projective over A it follows that $E \otimes_A \Omega$ is finite projective over Ω and therefore the trace $\int \colon \Omega \to \mathbb{C}$ extends to a trace, denoted again by $\int$, on $\mathrm{End}_A(E) \otimes \Omega$. The following result of Connes relates the value of the pairing as defined above to its value computed through the Chern–Weil formalism:

$$\boxed{\langle [\varphi],\, [\mathcal{E}] \rangle = \frac{1}{n!} \int \hat{\nabla}^{2n}}$$

The next example is a concrete illustration of this method.

Example 0.18.6. Let $\mathcal{E} = \mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing functions on the real line. The operators

$$(\xi \cdot U)(x) = \xi(x + \theta), \quad (\xi \cdot V)(x) = e^{2\pi i x}\xi(x)$$

turn $\mathcal{S}(\mathbb{R})$ into a right $\mathcal{A}_\theta$-module for all $\xi \in \mathcal{S}(\mathbb{R})$. It is the simplest of a series of modules $\mathcal{E}_{p,q}$ on the noncommutative torus, defined by Connes in [16]. It turns out that $\mathcal{E}$ is finite projective, and for the canonical trace τ on $\mathcal{A}_\theta$ we have

$$\langle \tau,\, \mathcal{E} \rangle = -\theta.$$

In Example 0.13.4 we defined a differential calculus, in fact a 2-cycle, on $\mathcal{A}_\theta$. It is easy to see that a connection on $\nabla \colon \mathcal{E} \to E \otimes_A \Omega^1$ with respect to this calculus is simply given by a pair of operators $\nabla_1, \nabla_2 \colon \mathcal{E} \to \mathcal{E}$ ('covariant derivatives' with respect to noncommutative vector fields δ_1 and δ_2) satisfying

$$\nabla_j(\xi a) = (\nabla_j \xi)a + \xi \delta_j(a), \quad j = 0, 1,$$

for all $\xi \in \mathcal{E}$ and $a \in \mathcal{A}_\theta$.

One can now check that the following formulae define a connection on $\mathcal{E}$ [16], [21]:

$$\nabla_1(\xi)(s) = -\frac{s}{\theta}\,\xi(s), \quad \nabla_2(\xi)(s) = \frac{d\xi}{ds}(s).$$

The curvature of this connection is constant and is given by

$$\nabla^2 = [\nabla_1,\, \nabla_2] = \frac{1}{\theta}\,\mathrm{id} \in \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}).$$

Exercise 0.18.1. Let E be a finite projective right A-module. Show that $\mathrm{End}_A(E) \simeq E \otimes_A E^*$ where $A^* = \mathrm{Hom}_A(E, A)$. The canonical pairing $E \otimes_A E^* \to A$ defined by $\xi \otimes f \mapsto f(\xi)$ induces a map $\mathrm{End}_A(E) \to A/[A, A] = HC_0(A)$. In particular if $\tau\colon A \to \mathbb{C}$ is a trace on A, the induced trace on $\mathrm{End}_A(E)$ is simply obtained by composing τ with the canonical pairing between E and E^*.

Exercise 0.18.2. Verify that under the natural quasi-isomorphism between (b, B) and cyclic complexes, formula (43) corresponds to (42).

Exercise 0.18.3. Show that a right A-module E admits a connection with respect to the universal differential calculus $(\Omega A,\, d)$, if and only if E is projective.

0.19 Connes–Chern character in K-homology

By K-homology for spaces we mean the theory which is dual to topological K-theory. While such a theory can be constructed using general techniques of algebraic topology, a beautiful and novel idea of Atiyah [2] (in the even case), and Brown–Douglas–Fillmore [10] (in the odd case) was to use techniques of index theory, functional analysis and operator algebras to define a K-homology theory (cf. also [4]). What is even more interesting is that the resulting theory can be extended to noncommutative algebras and pairs with K-theory. This extension, in full generality, then paved the way for Kasparov's bivariant KK-theory which unifies both K-theory and K-homology into a single bivariant theory (cf. [50] and [7]). Unfortunately the name K-homology is used even when one is dealing with algebras, despite the fact that the resulting functor is in fact contravariant for algebras, while K-theory for algebras is covariant. We hope this will cause no confusion for the reader.

To motivate the discussions, we start this section by recalling the notion of an *abstract elliptic operator* over a compact space [2]. This will then be extended to the noncommutative setting by introducing the notion of an, even or odd, Fredholm module over an algebra [19]. The Connes–Chern character of a Fredholm module is introduced next. We shall then define the index pairing between K-theory and K-homology, which indicates the sense in which these theories are dual to each other. The final result of this section is an index formula of Connes which computes the index pairing in terms of Connes–Chern characters for K-theory and K-homology.

Let X be a compact Hausdorff space. The even cycles for Atiyah's theory in [2] are *abstract elliptic operators* $(H,\, F)$ over $C(X)$. This means that $H = H^+ \oplus H^-$ is a $\mathbb{Z}_2$-graded Hilbert space, $\pi\colon C(X) \to \mathcal{L}(H)$ with

$$\pi(a) = \begin{pmatrix} \pi^+(a) & 0 \\ 0 & \pi^-(a) \end{pmatrix}$$

is an *even* representation of $C(X)$ in the algebra of bounded operators on H, and $F\colon H \to H$ with

$$F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \tag{47}$$

is an *odd* bounded operator with $F^2 - I \in \mathcal{K}(H)$ being compact. This data must satisfy the crucial condition

$$[F,\, \pi(a)] \in \mathcal{K}(H)$$

for all $a \in C(X)$. We shall make no attempt at turning these cycles into a homology theory. Suffice it to say that the homology theory is defined as the quotient of the set of these cycles by a homotopy equivalence relation (cf. [44], [32] for a recent account).

When X is a smooth closed manifold, the main examples of abstract elliptic operators in the above sense are given by elliptic pseudodifferential operators of order 0, $D\colon C^\infty(E^+) \to C^\infty(E^-)$ acting between sections of vector bundles E^+ and E^- on X. Let $P\colon H^+ \to H^-$ denote the natural extension of D to a bounded operator where $H^+ = L^2(E^+)$ and $H^- = L^2(E^-)$, and let $Q\colon H^- \to H^+$ denote a parametrix of P. Define F by (47). Then with $C(X)$ acting as multiplication operators on H^+ and H^-, basic elliptic theory shows that $(H,\, F)$ is an elliptic operator in the above sense on $C(X)$.

If $e \in M_n(C(X))$ is an idempotent representing a vector bundle on X, then the formula

$$\langle (H,\, F)\,,\, [e] \rangle := \operatorname{index} F_e^+$$

with the Fredholm operator $F_e^+ := eFe\colon eH^+ \to eH^-$ can be shown to define a pairing between the K-theory of X and abstract elliptic operators on X. This is the duality between K-homology and K-theory.

A modification of the above notion of abstract elliptic operator, which makes sense over noncommutative algebras, both in the even and odd case, is the following notion of Fredholm module in [19] which is an important variation of a related notion from [2], [10], [51] (cf. the remark below).

Definition 0.19.1. An *odd Fredholm module* over an algebra A is a pair $(H,\, F)$ where

1) H is a Hilbert space endowed with a representation

$$\pi\colon A \to \mathcal{L}(H);$$

2) $F \in \mathcal{L}(H)$ is a bounded selfadjoint operator with $F^2 = I$;

3) for all $a \in A$ we have

$$[F,\, \pi(a)] = F\pi(a) - \pi(a)F \in \mathcal{K}(H). \tag{48}$$

For $1 \le p < \infty$, let $\mathcal{L}^p(H)$ denote the Schatten ideal of p-summable operators. A Fredholm module $(H,\, F)$ is called *p-summable* if, instead of (48), we have the stronger condition:

$$[F,\, \pi(a)] \in \mathcal{L}^p(H) \tag{49}$$

for all $a \in A$. Since $\mathcal{L}^p(H) \subset \mathcal{L}^q(H)$ for $p \le q$, a p-summable Fredholm module is clearly q-summable for all $q \ge p$.

Definition 0.19.2. An *even Fredholm module* over an algebra A is a triple $(H,\, F,\, \gamma)$ such that $(H,\, F)$ is a Fredholm module over A in the sense of the above definition and $\gamma\colon H \to H$ is a bounded selfadjoint operator with $\gamma^2 = I$ and such that

$$F\gamma = -\gamma F, \quad \pi(a)\gamma = \gamma\pi(a) \tag{50}$$

for all $a \in A$.

Let H^+ and H^- denote the $+1$ and -1 eigenspaces of γ. They define an orthogonal decomposition $H = H^+ \oplus H^-$. With respect to this decomposition, equations (50) are equivalent to saying that π is an even representation and F is an odd operator, so that we can write

$$\pi(a) = \begin{pmatrix} \pi^+(a) & 0 \\ 0 & \pi^-(a) \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix},$$

where π^+ and π^- are representations of A on H^+ and H^-, respectively. The notion of a p-summable even Fredholm module is defined as in the odd case above.

Remark 4. Notice that in the preceding example with F defined by (47), and in general in [2], [10], [51], the condition $F^2 = I$ for a Fredholm module only holds modulo compact operators. Similarly for the two equalities in equation (50). It is shown in [21] that with simple modifications one can replace such (H, F) by an equivalent Fredholm module in which these equations, except (48), hold exactly. The point is that as far as pairing with K-theory is concerned the set up in [2], [10], [51] is enough. It is for the definition of the Chern character and pairing with cyclic cohomology that one needs the exact equalities $F^2 = I$ and (50), as well as the finite summability assumption (49).

Let $(H,\, F)$ be an odd p-summable Fredholm module over an algebra A and let n be an integer such that $2n \geq p$. To simplify the notation, from now on in our formulae the operator $\pi(a)$ will be denoted by a. Thus an expression like $a_0[F,\, a_1]$ stands for the operator $\pi(a_0)[F,\, \pi(a_1)]$, etc. We define a $(2n-1)$-cochain on A by

$$\boxed{\varphi_{2n-1}(a_0, a_1, \ldots, a_{2n-1}) = \mathrm{Tr}(F[F,\, a_0][F,\, a_1] \ldots [F,\, a_{2n-1}])} \tag{51}$$

where Tr denotes the operator trace. Notice that by our p-summability assumption, each commutator is in $\mathcal{L}^p(H)$ and hence, by Hölder inequality for Schatten class operators (cf. Appendix B), their product is in fact a trace class operator as soon as $2n \geq p$.

Proposition 0.19.1. *φ_{2n-1} is a cyclic $(2n-1)$-cocycle on A.*

Proof. For $a \in A$, let $da := [F,\, a]$. The following relations are easily established: for all $a, b \in A$ we have

$$d(ab) = ad(b) + da \cdot b \quad \text{and} \quad F da = -da \cdot F \tag{52}$$

Notice that for the second relation the assumption $F^2 = 1$ is essential. Now φ_{2n-1} can be written as

$$\varphi_{2n-1}(a_0, a_1, \ldots, a_{2n-1}) = \mathrm{Tr}(F da_0 da_1 \ldots da_{2n-1}),$$

and therefore

$$\begin{aligned}(b\varphi_{2n-1})(a_0, \ldots, a_{2n}) = \mathrm{Tr}\Big(\sum_{i=0}^{2n}(-1)^i F da_0 \ldots d(a_i da_{i+1}) \ldots da_{2n}\Big)\\ + (-1)^{2n+1}\,\mathrm{Tr}(Fd(a_{2n}a_0)da_1 \ldots da_{2n-1}).\end{aligned}$$

Using the derivation property of d, we see that most of the terms cancel and we are left with just four terms

$$\begin{aligned}&= \mathrm{Tr}(Fa_0 da_1 \ldots da_{2n}) - \mathrm{Tr}(F da_0 \ldots da_{2n-1}a_{2n})\\ &\quad + \mathrm{Tr}(Fa_{2n}da_0 \ldots da_{2n-1}) + \mathrm{Tr}(F da_{2n}a_0 da_1 \ldots da_{2n-1}).\end{aligned}$$

Using the relation $F da = -da \cdot F$ and the trace property of Tr we see that the second and third terms cancel. By the same argument the first and last terms cancel as well. This shows that φ_{2n-1} is a Hochschild cocycle.

To check the cyclic property of φ_{2n-1}, again using the relation $F da = -da \cdot F$, and the trace property of Tr, we have

$$\begin{aligned}\varphi_{2n-1}(a_{2n-1}, a_0, \ldots, a_{2n-2}) &= \mathrm{Tr}(F da_{2n-1}da_0 \ldots da_{2n-2})\\ &= -\,\mathrm{Tr}(da_{2n-1}F da_0 \ldots da_{2n-2})\\ &= -\,\mathrm{Tr}(F da_0 \ldots da_{2n-2}da_{2n-1})\\ &= -\varphi_{2n-1}(a_0, \ldots, a_{2n-2}, a_{2n-1}).\end{aligned}$$

□

Notice that if $2n \geq p-1$, then the cyclic cocycle (51) can be written as

$$\begin{aligned}\varphi_{2n-1}(a_0, a_1, \ldots, a_{2n-1}) &= 2\,\mathrm{Tr}(a_0[F,\, a_1] \ldots [F,\, a_{2n-1}])\\ &= 2\,\mathrm{Tr}(a_0 da_1 \ldots da_{2n-1})\end{aligned}$$

which looks remarkably like a noncommutative analogue of the integral

$$\int_M f_0\, df_1 \ldots df_{2n-1}.$$

Now the products $[F,\, a_0][F,\, a_1] \ldots [F,\, a_{2m-1}]$ are trace class for all $m \geq n$. Therefore we obtain a sequence of odd cyclic cocycles

$$\varphi_{2m-1}(a_0, a_1, \ldots, a_{2m-1}) = \mathrm{Tr}(F[F, a_0][F, a_1] \ldots [F, a_{2m-1}]), \quad m \geq n.$$

The next proposition shows that these cyclic cocycles are related to each other via the periodicity S-operator of cyclic cohomology:

Proposition 0.19.2. *For all $m \geq n$ we have*

$$S\varphi_{2m-1} = -\left(m + \frac{1}{2}\right)\varphi_{2m+1}.$$

Proof. Let us define a $2m$-cochain ψ_{2m} on A by the formula

$$\psi_{2m}(a_0, a_1, \dots, a_{2m}) = \mathrm{Tr}(F a_0 da_1 \dots da_{2m}).$$

We claim that

$$B\psi_{2m} = (2m)\,\varphi_{2m-1} \quad \text{and} \quad b\psi_{2m} = -\frac{1}{2}\,\varphi_{2m+1}.$$

In fact, we have $B\psi_{2m} = NB_0\psi_{2m}$, where

$$\begin{aligned} B_0\psi_{2m}(a_0, \dots, a_{2m-1}) &= \psi_{2m}(1, a_0, \dots, a_{2m-1}) \\ &\quad - (-1)^{2m-1}\psi_{2m}(a_0, \dots, a_{2m-1}, 1) \\ &= \mathrm{Tr}(F da_0 \dots da_{2m-1}), \end{aligned}$$

and hence

$$\begin{aligned} (NB_0)\psi_{2m}(a_0, \dots, a_{2m-1}) &= \mathrm{Tr}(F da_0 \dots da_{2m-1}) - \mathrm{Tr}(F da_{2m-1} da_0 \dots da_{2m-2}) \\ &\quad + \cdots - \mathrm{Tr}(F da_1 \dots da_0) \\ &= (2m)\,\mathrm{Tr}(F da_0 \dots da_{2m-1}) \\ &= (2m)\varphi_{2m-1}(a_0, \dots, a_{2m-1}) \end{aligned}$$

and

$$\begin{aligned} (b\psi_{2m})(a_0, \dots, a_{2m+1}) &= \mathrm{Tr}(F a_0 a_1 da_2 \dots da_{2m+1}) - \mathrm{Tr}(F a_0 d(a_1 a_2) \dots da_{2m+1}) \\ &\quad + \cdots + \mathrm{Tr}(F a_0 da_1 \dots d(a_{2m} a_{2m+1})) \\ &\quad - \mathrm{Tr}(F a_{2m+1} a_0 da_1 \dots da_{2m}). \end{aligned}$$

After cancelations, only two terms remain which can be collected into a single term:

$$\begin{aligned} &= \mathrm{Tr}(F a_0 da_1 \dots da_{2m} \cdot a_{2m+1}) - \mathrm{Tr}(F a_{2m+1} a_0 da_1 \dots da_{2m}) \\ &= -\frac{1}{2}\,\mathrm{Tr}(F da_0 da_1 \dots da_{2m+1}) \\ &= -\frac{1}{2}\varphi_{2m+1}(a_0, \dots, a_{2m+1}). \end{aligned}$$

The above computation shows that

$$bB^{-1}\varphi_{2m-1} = -\frac{1}{2(2m)}\varphi_{2m+1}.$$

Now using formula (31) for the operator S, we have

$$S\varphi_{2m-1} = (2m)(2m+1)bB^{-1}\varphi_{2m-1} = -\left(m + \frac{1}{2}\right)\varphi_{2m+1}. \qquad \square$$

The odd Connes–Chern characters $\mathrm{Ch}^{2m-1} = \mathrm{Ch}^{2m-1}(H, F)$, are defined by rescaling the cocycles φ_{2m-1} appropriately. Let

$$\begin{aligned}\mathrm{Ch}^{2m-1}&(a_0, \dots, a_{2m-1})\\ &:= (-1)^m 2\left(m-\frac{1}{2}\right)\dots\frac{1}{2}\ \mathrm{Tr}(F[F,a_0][F,a_1]\dots[F,a_{2m-1}]).\end{aligned}$$

The following is an immediate corollary of the above proposition:

Corollary 0.19.1. *We have*

$$S(\mathrm{Ch}^{2m-1}) = \mathrm{Ch}^{2m+1} \quad \textit{for all } m \geq n.$$

Definition 0.19.3. The *Connes–Chern character* of an odd p-summable Fredholm module (H, F) over an algebra A is the class of the cyclic cocycle Ch^{2m-1} in the odd periodic cyclic cohomology group $HP^{\mathrm{odd}}(A)$.

By the above corollary, the class of Ch^{2m-1} in $HP^{\mathrm{odd}}(A)$ is independent of the choice of m.

Example 0.19.1. Let $A = C(S^1)$ ($S^1 = \mathbb{R}/\mathbb{Z}$) act on $H = L^2(S^1)$ as multiplication operators. Let $F(e_n) = e_n$ if $n \geq 0$ and $F(e_n) = -e_n$ for $n < 0$, where $e_n(x) = e^{2\pi i n x}$, $n \in \mathbb{Z}$, denotes the standard basis of H. Clearly F is selfadjoint and $F^2 = I$. To show that $[F, \pi(f)]$ is a compact operator for all $f \in C(S^1)$, notice that if $f = \sum_{|n| \leq N} a_n e_n$ is a finite trigonometric sum then $[F, \pi(f)]$ is a finite rank operator and hence is compact. In general we can uniformly approximate a continuous function by a trigonometric sum and show that the commutator is compact for any continuous f. This shows that (H, F) is an odd Fredholm module over $C(S^1)$.

This Fredholm module is not p-summable for any $1 \leq p < \infty$. If we restrict it to the subalgebra $C^\infty(S^1)$ of smooth functions, then it can be checked that (H, F) is in fact p-summable for all $p > 1$, but is not 1-summable even in this case.

Let us compute the Chern character of this Fredholm module with $A = C^\infty(S^1)$. By the above definition, $\mathrm{Ch}^1(H, F) = [\varphi_1]$ is the class of the following cyclic 1-cocycle in $HP^{\mathrm{odd}}(A)$:

$$\varphi_1(f_0, f_1) = \mathrm{Tr}(F[F, f_0][F, f_1]),$$

and the question is if we can identify this cocycle with some local formula. We claim that

$$\mathrm{Tr}(F[F, f_0][F, f_1]) = \frac{4}{2\pi i}\int f_0\, df_1 \quad \text{for all } f_0, f_1 \in A.$$

To verify the claim, it suffices to check it for the basis elements $f_0 = e_m, f_1 = e_n$ for all $m, n \in \mathbb{Z}$. The right-hand side is easily computed:

$$\frac{4}{2\pi i}\int e_m\, de_n = \begin{cases} 0 & \text{if } m+n \neq 0, \\ 4n & \text{if } m+n = 0. \end{cases}$$

To compute the left-hand side, notice that

$$[F,\, e_n](e_k) = \begin{cases} 0 & \text{if } k \geq 0,\, n+k \geq 0, \\ -2e_{n+k} & \text{if } k \geq 0,\, n+k < 0, \\ 2e_{n+k} & \text{if } k < 0,\, n+k \geq 0, \\ 0 & \text{if } k < 0,\, n+k < 0. \end{cases}$$

From this we conclude that $Fe_m[F, e_n] = 0$ if $m+n \neq 0$. To compute the operator trace for $m + n = 0$, we use the formula $\mathrm{Tr}(T) = \sum_k \langle T(e_k), e_k \rangle$ for the trace of a trace class operator T. Using the above information we have

$$F[F,\, e_{-n}][F,\, e_n](e_k) = \begin{cases} 0 & \text{if } k \geq 0,\, n+k \geq 0, \\ 4e_k & \text{if } k \geq 0,\, n+k < 0, \\ 4e_k & \text{if } k < 0,\, n+k \geq 0, \\ 0 & \text{if } k < 0,\, n+k < 0, \end{cases}$$

from which we readily obtain

$$\mathrm{Tr}(F\,[F,\, e_{-n}][F,\, e_n]) = 4n \quad \text{for all } n \in \mathbb{Z}.$$

This finishes the proof.

Next we turn to the even case. Let $(H,\, F,\, \gamma)$ be an even p-summable Fredholm module over an algebra A and let n be an integer such that $2n + 1 \geq p$. Define a $2n$-cochain on A by the formula

$$\varphi_{2n}(a_0, a_1, \ldots, a_{2n}) = \mathrm{Tr}(\gamma F[F, a_0][F, a_1] \ldots [F, a_{2n}]). \tag{53}$$

Proposition 0.19.3. *φ_{2n} is a cyclic $2n$-cocycle on A.*

Proof. The proof is similar to the odd case and is left to the reader. Apart from relations (52), one needs the auxiliary relation $\gamma\, da = -da\, \gamma$ for the proof as well. $\square$

Notice that if we have the stronger condition $2n \geq p$, then the cyclic cocycle (53) can be written as

$$\begin{aligned} \varphi_{2n}(a_0, a_1, \ldots, a_{2n}) &= \mathrm{Tr}(\gamma a_0[F, a_1] \ldots [F, a_{2n}]) \\ &= \mathrm{Tr}(\gamma a_0 da_1 \ldots da_{2n}). \end{aligned}$$

As in the odd case, we obtain a sequence of even cyclic cocycles φ_{2m}, defined by

$$\varphi_{2m}(a_0, a_1, \ldots, a_{2m}) = \mathrm{Tr}(\gamma F[F, a_0][F, a_1] \ldots [F, a_{2m}]), \quad m \geq n.$$

Proposition 0.19.4. *For all $m \geq n$ we have*

$$S\varphi_{2m} = -(m+1)\varphi_{2m+2}.$$

Proof. Define a $(2m+1)$-cochain ψ_{2m+1} on A by the formula

$$\psi_{2m+1}(a_0, a_1, \ldots, a_{2m+1}) = \mathrm{Tr}(\gamma\, F a_0 da_1 \ldots da_{2m+1}).$$

The following relations can be proved as in the odd case:

$$B\psi_{2m+1} = (2m+1)\varphi_{2m} \quad \text{and} \quad b\psi_{2m+1} = -\frac{1}{2}\varphi_{2m+2}.$$

It shows that

$$bB^{-1}\varphi_{2m} = -\frac{1}{2(2m+1)}\varphi_{2m+2},$$

so that using formula (31) for the operator S we obtain

$$S\varphi_{2m} = (2m+1)(2m+2)bB^{-1}\varphi_{2m} = -(m+1)\varphi_{2m+2}.$$ □

The even Connes–Chern characters $\mathrm{Ch}^{2m} = \mathrm{Ch}^{2m}(H, F, \gamma)$ are now defined by rescaling the even cyclic cocycles φ_{2m}:

$$\boxed{\mathrm{Ch}^{2m}(a_0, a_1, \ldots, a_{2m}) := \tfrac{(-1)^m m!}{2}\ \mathrm{Tr}(\gamma F[F, a_0][F, a_1]\ldots[F, a_{2m}])} \tag{54}$$

The following is an immediate corollary of the above proposition:

Corollary 0.19.2. *We have*

$$S(\mathrm{Ch}^{2m}) = \mathrm{Ch}^{2m+2} \quad \textit{for all } m \geq n.$$

Definition 0.19.4. The *Connes–Chern character* of an even p-summable Fredholm module $(H,\ F, \gamma)$ over an algebra A is the class of the cyclic cocycle $\mathrm{Ch}^{2m}(H,\ F,\ \gamma)$ in the even periodic cyclic cohomology group $HP^{\mathrm{even}}(A)$.

By the above corollary, the class of Ch^{2m} in $HP^{\mathrm{even}}(A)$ is independent of m.

Example 0.19.2 (A noncommutative example)**.** Following [19], we construct an even Fredholm module over $A = C_r^*(F_2)$, the reduced group C^*-algebra of the free group on two generators. This Fredholm module is not p-summable for any p, but by restricting it to a properly defined dense subalgebra of A (which plays the role of 'smooth functions' on the underlying noncommutative space), we shall obtain a 1-summable Fredholm. We shall also identify the character of this 1-summable module. It is known that a group is free if and only if it has a free action on a tree. Let then T be a tree with a free action of F_2, and let T^0 and T^1 denote the set of vertices and 1-simplices of T, respectively. Let

$$H^+ = \ell^2(T^0) \quad \text{and} \quad H^- = \ell^2(T^1) \oplus \mathbb{C},$$

and let the canonical basis of $\ell^2(T^0)$ (resp. $\ell^2(T^1)$) be denoted by ε_q, $q \in T^0$ (resp. $q \in T^1$). Fixing a vertex $p \in T^0$, we can define a one-to-one correspondence

$$\varphi\colon T^0 - \{p\} \to T^1$$

by sending $q \in T^0 - \{p\}$ to the unique 1-simplex containing q and lying between p and q. This defines a unitary operator $P\colon H^+ \to H^-$ by

$$P(\varepsilon_q) = \varepsilon_{\varphi(q)} \quad \text{if } q \neq p, \quad \text{and} \quad P(\varepsilon_p) = (0, 1).$$

The action of F_2 on T^0 and T^1 induces representations of $C_r^*(F_2)$ on $\ell^2(T^0)$ and $\ell^2(T^1)$ and on $H^- = \ell^2(T^1) \oplus \mathbb{C}$ by the formula $a(\xi, \lambda) = (a\xi, 0)$. Let $H = H^+ \oplus H^-$ and

$$F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To check that (H, F, γ) is a Fredholm module over A we need to verify that $[F, a] \in \mathcal{K}(H)$ for all $a \in C_r^*(F_2)$. Since the group algebra $\mathbb{C}F_2$ is dense in $C_r^*(F_2)$, it suffices to check that for all $a = g \in F_2$, the commutator $[F, g]$ is a finite rank operator. This in turn is a consequence of the easily established fact that for all $g \in F_2$,

$$\varphi(gq) = g\varphi(q) \quad \text{for all } q \neq g^{-1}p.$$

In fact, for $q \in T^0$ we have

$$[F, g](\varepsilon_q) = F(\varepsilon_{gq}) - gF(\varepsilon_q) = \varepsilon_{\varphi(gq)} - \varepsilon_{g\varphi(q)} = 0$$

if $q \neq g^{-1}p$, and $[F, g](\varepsilon_{g^{-1}p}) = (0, 1) - g\varepsilon_{\varphi(g^{-1}p)}$. A similar argument works for the basis elements ε_q, $q \in T^1$. This shows that $[F, g]$ is a rank one operator.

Let

$$\mathcal{A} = \{a \in A;\ [F, a] \in \mathcal{L}^1(H)\}.$$

Using the relation $[F, ab] = a[F, b] + [F, a]b$, it is clear that $\mathcal{A}$ is a subalgebra of A. It is also dense in A as it contains the group algebra $\mathbb{C}F_2$. Though we do not need it now, it can also be shown that $\mathcal{A}$ is *stable under holomorphic functional calculus* and in particular the inclusion $\mathcal{A} \subset A$ induces an isomorphism $K_0(\mathcal{A}) \to K_0(A)$ in K-theory (cf. Section 4.3 for more on this). By its very definition, we now have an even 1-summable Fredholm module (H, F, γ) over $\mathcal{A}$ and it remains to compute its character. Let $\tau\colon A \to \mathbb{C}$ denote the canonical trace on A. We claim that

$$\frac{1}{2}\ \mathrm{Tr}(\gamma F[F, a]) = \tau(a) \quad \text{for all } a \in \mathcal{A}, \tag{55}$$

so that

$$\mathrm{Ch}(H, F, \gamma)(a) = \tau(a).$$

To verify the claim, notice that

$$\gamma F[F, a] = \begin{bmatrix} a - P^{-1}aP & 0 \\ 0 & -a + PaP^{-1} \end{bmatrix},$$

so that

$$\frac{1}{2}\ \mathrm{Tr}(\gamma F[F, a]) = \mathrm{Tr}(a - P^{-1}aP).$$

Now for the operator $a - P^{-1}aP\colon H^+ \to H^+$ we have

$$\langle a\varepsilon_q,\, \varepsilon_q\rangle = \tau(a) \quad \text{for all } q \in T^0,$$

and

$$\langle (P^{-1}aP)(\varepsilon_p),\, \varepsilon_p\rangle = 0 \quad \text{and} \quad \langle (P^{-1}aP)(\varepsilon_q),\, \varepsilon_q\rangle = \tau(a)$$

for all $q \neq p$, from which (55) follows.

Our next goal is to define the *index pairing* between Fredholm modules over A and the K-theory of A. Notice that for this we do not need to assume that the Fredholm module is finitely summable. We start with the even case. Let $(H,\, F,\, \gamma)$ be an even Fredholm module over an algebra A and let $e \in A$ be an idempotent. Let

$$F_e^+\colon eH_0 \to eH_1$$

denote the restriction of the operator eFe to the subspace eH_0. It is a Fredholm operator. To see this, let $F_e^-\colon eH_1 \to eH_0$ denote the restriction of eFe to the subspace eH_1. We claim that the operators $F_e^+F_e^- - 1$ and $F_e^-F_e^+ - 1$ are compact. Atkinson's theorem then shows that F_e^+ is Fredholm. The claim follows from the following computation:

$$eFeeFe = e(Fe - eF + eF)Fe = e[F,e]Fe + e,$$

and the fact that $[F,e]$ is a compact operator.

For an idempotent $e \in A$ let us define a pairing:

$$\boxed{\langle (H,\, F, \gamma),\, [e]\rangle := \operatorname{index} F_e^+}$$

More generally, if $e \in M_n(A)$ is an idempotent in the algebra of n by n matrices over A, we define

$$\langle (H,\, F, \gamma),\, [e]\rangle := \langle (H^n,\, F^n, \gamma),\, [e]\rangle,$$

where $(H^n,\, F^n)$ is the *n-fold inflation* of (H, F) defined by $H^n = H \otimes \mathbb{C}^n$, $F^n = F \otimes I_{\mathbb{C}^n}$. It is easily seen that $(H^n,\, F^n)$ is a Fredholm module over $M_n(A)$ and if (H, F) is p-summable then so is $(H^n,\, F^n)$. It is easily checked that the resulting map is additive with respect to direct sum of idempotents and is conjugation invariant. This shows that each even Fredholm module, which need not be finitely summable, induces an additive map on K-theory:

$$\langle (H,\, F,\, \gamma),\, -\rangle\colon K_0(A) \to \mathbb{C}.$$

There is a similar index pairing between odd Fredholm modules over A and the algebraic K-theory group $K_1^{\text{alg}}(A)$. Let $(H,\, F)$ be an odd Fredholm module over A and let $U \in A^\times$ be an invertible element in A. Let $P = \frac{F+1}{2}\colon H \to H$ be the projection operator defined by F. Let us check that the operator

$$PUP\colon PH \to PH$$

is a Fredholm operator. Again the proof hinges on Atkinson's theorem and noticing that $PU^{-1}P$ is an inverse for PUP modulo compact operators. We have

$$PUPPU^{-1}P - I_{PH} = PUPU^{-1}P - I_{PH} = P(UP - PU + PU)U^{-1}P - I_{PH}$$
$$= P[U,\, P]U^{-1}P + P - I_{PH} = \frac{1}{2}P[U,\, F]U^{-1}P.$$

But $[F, U]$ is a compact operator by our definition of Fredholm modules and hence the last term is compact too. Similarly one checks that $PU^{-1}PPUP - I_{PH}$ is a compact operator as well. We can thus define the index pairing:

$$\boxed{\langle (H,\, F),\, [U]\rangle := \operatorname{index}(PUP)}$$

If the invertible U happens to be in $M_n(A)$ we can apply the above definition to the n-fold iteration of $(H,\, F)$, as in the even case above, to define the pairing. The resulting map can be shown to induce a well-defined additive map

$$\langle (H,\, F),\, -\rangle \colon K_1^{\mathrm{alg}}(A) \to \mathbb{C}.$$

Example 0.19.3. Let $(H,\, F)$ be the Fredholm module of Example **??** and let $f \in C(S^1)$ be a nowhere zero continuous function on S^1 representing an element of $K_1^{\mathrm{alg}}(C(S^1))$. We want to compute the index pairing $\langle [(H,\, F)],\, [f]\rangle = \operatorname{index}(PfP)$. The operator $PfP \colon H^+ \to H^+$ is called a *Toeplitz operator*. The following standard result, known as the Gohberg–Krein index theorem, computes the index of a Toeplitz operator in terms of the winding number of f:

$$\langle [(H,\, F)],\, [f]\rangle = \operatorname{index}(PfP) = -W(f, 0).$$

To prove this formula notice that both sides are homotopy invariant. For the left-hand side this is a consequence of the homotopy invariance of the Fredholm index while for the right-hand side it is a standard fact about the winding number. Also, both sides are additive. Therefore it suffices to show that the two sides coincide on the generator of $\pi_1(S^1)$, i.e., for $f(z) = z$. Then PzP is easily seen to be the forward shift operator given by $PzP(e_n) = e_{n+1}$ in the given basis. Clearly then $\operatorname{index}(PzP) = -1 = -W(z, 0)$.

When f is smooth we have the following well-known formula for the winding number:

$$W(f, 0) = \frac{1}{2\pi i}\int f^{-1}\, df = \frac{1}{2\pi i}\varphi(f^{-1}, f),$$

where φ is the cyclic 1-cocycle on $C^\infty(S^1)$ defined by $\varphi(f, g) = \int f\, dg$. Since this cyclic cocycle is the Connes–Chern character of the Fredholm module (H, F), the above equation can be written as

$$\langle [(H,\, F)],\, [f]\rangle = \frac{1}{2\pi i}\langle \mathrm{Ch}^{\mathrm{odd}}(H, F),\, \mathrm{Ch}_{\mathrm{odd}}(f)\rangle,$$

where the pairing on the right-hand side is between cyclic cohomology and homology. As we shall prove next, this is a special case of a very general index formula of Connes.

0.20 Connes' index theorem

Now what makes the Connes–Chern character in K-homology useful is the fact that it can capture the analytic index by giving a topological formula for the index. More precisely we have the following index formula due to Connes [19]:

Theorem 0.20.1. *Let (H, F, γ) be an even p-summable Fredholm module over A and n be an integer such that $2n+1 \geq p$. If e is an idempotent in A then*

$$\operatorname{index}(F_e^+) = \frac{(-1)^n}{2}\, \varphi_{2n}(e, e, \ldots, e),$$

where the cyclic $2n$-cocycle φ_{2n} is defined by

$$\varphi_{2n}(a_0, a_1, \ldots, a_{2n}) = \operatorname{Tr}(\gamma F[F, a_0][F, a_1] \ldots [F, a_{2n}]).$$

Proof. We use the following fact from the theory of Fredholm operators (cf. Proposition A.2 for a proof): let $P' \colon H' \to H''$ be a Fredholm operator and let $Q' \colon H'' \to H'$ be such that for an integer $n \geq 0$, $1 - P'Q' \in \mathcal{L}^{n+1}(H'')$ and $1 - Q'P' \in \mathcal{L}^{n+1}(H')$. Then

$$\operatorname{index}(P') = \operatorname{Tr}(1 - Q'P')^{n+1} - \operatorname{Tr}(1 - P'Q')^{n+1}.$$

We can also write the above formula as a supertrace

$$\operatorname{index}(P') = \operatorname{Tr}(\gamma'(1 - F'^2)^{n+1}), \tag{56}$$

where the operators F' and γ' acting on $H' \oplus H'$ are defined by

$$F' = \begin{pmatrix} 0 & Q' \\ P' & 0 \end{pmatrix} \quad \text{and} \quad \gamma' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We apply this result to $P' = F_e^+$, $Q' = F_e^-$, $H' = e_0 H_0$ and $H'' = e_1 H_1$. By our summability assumption, both operators $1 - P'Q'$ and $1 - Q'P'$ are in $\mathcal{L}^{n+1}$ and we can apply (56). We have

$$\operatorname{index}(F_e^+) = \operatorname{Tr}(\gamma'\,(1 - F'^2)^{n+1}) = \operatorname{Tr}(\gamma\,(e - (eFe)^2)^{n+1}).$$

As in the proof of Proposition 0.19.1 let $de := [F, e]$. Using the relations $e^2 = e$, $ede \cdot e = 0$, and $edede = de \cdot de \cdot e$, we have

$$e - (eFe)^2 = eF(Fe - eF)e = (eF - Fe + Fe)(Fe - eF)e = -edede$$

and hence $(e - (eFe)^2)^{n+1} = (-1)^{n+1}(edede)^{n+1} = (-1)^{n+1} e(de)^{2n+2}$. Thus the index can be written as

$$\operatorname{index}(F_e^+) = (-1)^{n+1} \operatorname{Tr}(\gamma e (de)^{2n+2}).$$

On the other hand, using $de = ede + de \cdot e$, we have

$$\begin{aligned}\varphi_{2n}(e, e, \dots, e) &= \operatorname{Tr}(\gamma F(de)^{2n+1})\\ &= \operatorname{Tr}(\gamma(Fe)e(de)^{2n+1} + \gamma F de \cdot ee(de)^{2n}\\ &= \operatorname{Tr}(Fe - eF + eF)e(de)^{2n+1}) + \gamma(eF - Fe + Fe)de \cdot e(de)^{2n}\\ &= \operatorname{Tr}(\gamma de \cdot e(de)^{2n+1} - \operatorname{Tr}(\gamma dede \cdot e(de)^{2n})\\ &= -\operatorname{Tr}(\gamma dede \cdot e(de)^{2n}) - \operatorname{Tr}(\gamma dede \cdot e(de)^{2n})\\ &= -2\operatorname{Tr}(\gamma e(de)^{2n+2})\end{aligned}$$

which of course proves the theorem. In the above computation we used the fact that $\operatorname{Tr}(\gamma eFe(de)^{2n+1}) = \operatorname{Tr}(\gamma Fede \cdot e(de)^{2n}) = 0$. □

Using the pairing $HC^{2n}(A) \otimes K^0(A) \to \mathbb{C}$ between cyclic cohomology and K-theory defined in (42), and the definition of the Connes–Chern character of (H, F, γ) in (54), the above index formula can be written as

$$\operatorname{index}(F_e^+) = \langle \operatorname{Ch}^{2n}(H,\, F,\, \gamma),\, [e]\rangle,$$

or in its stable form

$$\operatorname{index}(F_e^+) = \langle \operatorname{Ch}^{\text{even}}(H,\, F,\, \gamma),\, [e]\rangle.$$

There is yet another way to interpret the index formula as

$$\langle (H,\, F,\, \gamma),\, [e]\rangle = \langle \operatorname{Ch}^{2n}(H,\, F,\, \gamma),\, \operatorname{Ch}_{2n}[e]\rangle,$$

where on the left-hand side we have the pairing between K-homology and K-theory and on the right-hand side the pairing between cyclic cohomology and homology.

The corresponding index formula in the odd case is as follows:

Proposition 0.20.1. *Let $(H,\, F)$ be an odd p-summable Fredholm module over A and let n be an integer such that $2n \geq p$. If u is an invertible element in A, then*

$$\operatorname{index}(PuP) = \frac{(-1)^n}{2^{2n}} \varphi_{2n-1}(u^{-1}, u, \dots, u^{-1}, u),$$

where the cyclic cocycle φ_{2n-1} is defined by

$$\varphi_{2n-1}(a_0, a_1, \dots, a_{2n-1}) = \operatorname{Tr}(F[F, a_0][F, a_1] \dots [F, a_{2n-1}]).$$

Proof. Let $P = \frac{1+F}{2}$, $H' = PH$, $P' = PuP \colon H' \to H'$, $Q' = Pu^{-1}P \colon H' \to H'$, and $du = [F, u]$. We have

$$\begin{aligned} 1 - Q'P' &= 1 - Pu^{-1}PPuP \\ &= 1 - pu^{-1}(Pu - uP + uP)P \\ &= 1 - Pu^{-1}[P, u]P - P \\ &= -\frac{1}{2}Pu^{-1}du \cdot P \\ &= \frac{1}{2}Pdu^{-1} \cdot uP = \frac{1}{2}Pdu^{-1}(uP - Pu + Pu) \\ &= -\frac{1}{4}Pdu^{-1}du + Pdu^{-1}Pu \\ &= -\frac{1}{4}Pdu^{-1}du, \end{aligned}$$

where in the last step the relation $Pdu^{-1}P = 0$ was used. This relation follows from

$$[P,\ u^{-1}] = [P^2,\ u^{-1}] = P[P,\ u^{-1}] + [P,\ u^{-1}]P.$$

Since, by our summability assumption, $du = [F, u] \in \mathcal{L}^{2n}(H)$ and similarly $du^{-1} \in \mathcal{L}^{2n}(H)$, we have $1 - Q'P' \in \mathcal{L}^n(H')$.

A similar computation shows that

$$1 - P'Q' = -\frac{1}{4}Pdudu^{-1},$$

and hence $1 - P'Q' \in \mathcal{L}^n(H')$.

Using formula (56) for the index, we obtain

$$\begin{aligned} \operatorname{index}(PuP) &= \operatorname{Tr}((1 - Q'P')^n) - \operatorname{Tr}((1 - P'Q')^n) \\ &= \frac{(-1)^n}{2^{2n}} \operatorname{Tr}((Pdu^{-1}du)^n) - \frac{(-1)^n}{2^{2n}} \operatorname{Tr}((Pdudu^{-1})^n) \\ &= \frac{(-1)^n}{2^{2n}} \operatorname{Tr}(P(du^{-1}du)^n) - \frac{(-1)^n}{2^{2n}} \operatorname{Tr}(P(dudu^{-1})^n) \\ &= \frac{(-1)^n}{2^{2n}} \operatorname{Tr}\left(\frac{1+F}{2}\,(du^{-1}du)^n\right) - \frac{(-1)^n}{2^{2n}} \operatorname{Tr}\left(\frac{1+F}{2}\,(dudu^{-1})^n\right) \\ &= \frac{(-1)^n}{2^{2n}} \operatorname{Tr}(F\,(du^{-1}du)^n) \\ &= \frac{(-1)^n}{2^{2n}} \varphi_{2n-1}(u^{-1}, u, \ldots, u^{-1}, u), \end{aligned}$$

where in the last step we used the relations

$$\operatorname{Tr}((du^{-1}du)^n) = \operatorname{Tr}((dudu^{-1})^n) \quad \text{and} \quad \operatorname{Tr}(F(du^{-1}du)^n) = -\operatorname{Tr}(F(dudu^{-1})^n).$$

□

Using the pairing $HC^{2n-1}(A)\otimes K_1^{\text{alg}}(A) \to \mathbb{C}$ and the definition of $\text{Ch}^{2n-1}(H, F)$, the above index formula can be written as

$$\text{index}(PuP) = \langle \text{Ch}^{2n-1}(H, F), [u]\rangle,$$

or in its stable form

$$\text{index}(PuP) = \langle \text{Ch}^{\text{odd}}(H, F), [u]\rangle.$$

There is yet another way to interpret the index formula as

$$\langle (H, F), [u]\rangle = \langle \text{Ch}^{2n-1}(H, F), \text{Ch}_{2n-1}[u]\rangle,$$

where on the left-hand side we have the pairing between K-homology and K-theory and on the right-hand side the pairing between cyclic cohomology and homology.

Example 0.20.1 (A noncommutative connected space)**.** A *projection* in a $*$-algebra is an element e satisfying $e^2 = e = e^*$. It is called a trivial projection if $e = 0$ or $e = 1$. It is clear that a compact space X is connected if and only if the algebra $C(X)$ has no non-trivial projections. Let us agree to call a noncommutative space represented by a C^*-algebra A connected if A has no non-trivial projections. The *Kadison conjecture* states that the reduced group C^*-algebra of a torsion-free discrete group is connected. This conjecture, in its full generality, is still open although it has now been verified for various classes of groups [**?**]. Methods of noncommutative geometry play an important role in these proofs. The validity of the conjecture for free groups was first established by Pimsner and Voiculescu [**?**] using techniques of K-theory. Here we reproduce Connes' proof of this conjecture for free groups. Note that the conjecture is obviously true for the finitely generated free abelian groups $\mathbb{Z}^n$, since by Fourier theory, or the Gelfand–Naimark theorem, $C^*(\mathbb{Z}^n) \simeq C(\mathbb{T}^n)$, and the n-torus T^n is of course connected.

Let $\tau\colon C_r^*(F_2) \to \mathbb{C}$ be the canonical normalized trace. It is positive and faithful in the sense that for all $a \in A$, $\tau(aa^*) \geq 0$ and $\tau(aa^*) = 0$ if and only if $a = 0$. Thus if we can show that for a projection e, $\tau(e)$ is an integer then we can deduce that $e = 0$ or $e = 1$. In fact since e is a projection we have $0 \leq e \leq 1$ and therefore $0 \leq \tau(e) \leq 1$, and by integrality we have $\tau(e) = 1$ or $\tau(e) = 0$. Since τ is faithful from $0 = \tau(e) = \tau(ee^*)$ we have $e = 0$. A similar argument works for $\tau(e) = 1$.

Now the proof of the *integrality* of $\tau(e)$ is based on Connes' index formula in Theorem 0.20.1 and is remarkably similar to proofs of classical integrality theorems for characteristic numbers in topology using an index theorem: to show that a number $\tau(e)$ is an integer it suffices to shows that it is the index of a Fredholm operator. Let (H, F, γ) be the even 1-summable Fredholm module over the dense subalgebra $\mathcal{A} \subset C_r^*(F_2)$ defined in Example 0.19.2. The index formula combined with (55), shows that if $e \in \mathcal{A}$ is a projection then

$$\tau(e) = \frac{1}{2}\,\text{Tr}(\gamma F e[F, e]) = \text{index}(F_e^+) \in \mathbb{Z}$$

is an integer and we are done. To prove the integrality result for idempotents in A which are not necessarily in $\mathcal{A}$, we make use of the fact that $\mathcal{A}$ is stable under holomorphic functional calculus. Let $e \in A$ be an idempotent. For any $\epsilon > 0$ there is an idempotent $e' \in \mathcal{A}$ such that $\|e - e'\| < \epsilon$. In fact, since $\mathcal{A}$ is dense in A we can first approximate it by an element $g \in \mathcal{A}$. Since $\mathrm{sp}(e) \subset \{0, 1\}$, $\mathrm{sp}(g)$ is concentrated around 0 and 1. Let f be a holomorphic function defined on an open neighborhood of $\mathrm{sp}(g)$ which is identically equal to 0 around 0 and identically equal to 1 around 1. Then

$$e' = f(g) = \frac{1}{2\pi i} \int_\gamma f(z)(z1 - e)^{-1}\, dz,$$

is an idempotent in $\mathcal{A}$ which is close to e. As we showed before (cf. formula (B.1)), close idempotents are equivalent in the sense that $e = ue'u^{-1}$ for an appropriate $u \in A$. In particular we conclude that $\tau(e) = \tau(ue'u^{-1}) = \tau(e')$ is an integer.

In connection with Exercise 0.20.6 it is appropriate to mention that there is a refinement of the notion of Fredholm module to that of a *spectral triple* that plays a very important role in further developments of noncommutative geometry. Broadly speaking, going from Fredholm modules to spectral triples is like passing from the conformal class of a metric to the Riemannian metric itself. Spectral triples simultaneously provide a notion of *Dirac operator* in noncommutative geometry, as well as a Riemannian type *distance function* for noncommutative spaces.

To motivate the definition of a spectral triple, we recall that the Dirac operator $D\!\!\!/$ on a compact Riemannian spin^c manifold acts as an unbounded selfadjoint operator on the Hilbert space $L^2(M, S)$ of L^2-spinors on the manifold M. If we let $C^\infty(M)$ act on $L^2(M, S)$ by multiplication operators, then one can check that for any smooth function f, the commutator $[D, f] = Df - fD$ extends to a bounded operator in $L^2(M, S)$. Now the geodesic distance d on M can be recovered from the following *distance formula* of Connes [21]:

$$d(p, q) = \sup\{|f(p) - f(q)|;\, \|\, [D, f]\, \| \leq 1\} \quad \text{for all } p, q \in M. \tag{57}$$

The triple $(C^\infty(M), L^2(M, S), D\!\!\!/)$ is a commutative example of a spectral triple.

In general, in the odd case, a spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a $*$-algebra represented by bounded operators on a Hilbert space $\mathcal{H}$, and D, encoding the Dirac operator and metric, is an unbounded selfadjoint operator on $\mathcal{H}$. It is required that D interacts with the algebra in a bounded fashion, i.e., that for all $a \in \mathcal{A}$ the commutators $[D, a] = Da - aD$ are well defined on the domain of D and extend to bounded operators on $\mathcal{H}$. It is further postulated that the operator D should have *compact resolvent* in the sense that $(D + \lambda)^{-1} \in \mathcal{K}(\mathcal{H})$ for all $\lambda \notin \mathbb{R}$. This last condition implies that the spectrum of D consists of a discrete set of eigenvalues with finite multiplicity. A spectral triple is called finitely summable if $(D^2 + 1)^{-1} \in \mathcal{L}^p(\mathcal{H})$ for some $1 \leq p < \infty$.

Given a spectral triple as above and assuming that D is invertible, one checks that with $F := D|D|^{-1}$, the phase of D, $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module. By passing from the spectral triple to the corresponding Fredholm module, we lose the metric structure, but still retain the topological information, in particular the index pairing, encoded by the triple. For examples of spectral triples arising in physics and number theory the reader should consult [23].

Exercise 0.20.1. The Fredholm module of Example 0.19.2 can be defined over any free group. For $\Gamma = \mathbb{Z}$ one obtains an even Fredholm module over $C^*(\mathbb{Z}) \simeq C(S^1)$. Identify this Fredholm module and its character.

Exercise 0.20.2. Give an example of a discrete group Γ and a projection $e \in \mathbb{C}\Gamma$ such that $\tau(e)$ is not an integer (τ is the canonical trace).

Exercise 0.20.3. Let (H, F) be an odd p-summable Fredholm module over an algebra A. What happens if in the cochain (51) we replace $(2n-1)$ by an even integer. Similarly for even Fredholm modules.

Exercise 0.20.4. Show that the Fredholm module in Example 0.19.1 is p-summable for any $p > 1$ but is not 1-summable. If we consider it as a Fredholm module over the algebra $C^k(S^1)$ of k-times continuously differentiable functions then (H, F) is p-summable for some $p > 1$. Find a relation between k and p.

Exercise 0.20.5. Show that the Fredholm module over $C_r^*(F_2)$ in Example 0.19.2 is not 1-summable.

Exercise 0.20.6. Let $D = -i\frac{d}{dx}\colon C^\infty(S^1) \to C^\infty(S^1)$. It has an extension to a selfadjoint unbounded operator $D\colon \mathrm{Dom}(D) \subset L^2(S^1) \to L^2(S^1)$. Show that the arc distance on S^1 can be recovered from D via the formula

$$\mathrm{dist}(p, q) = \sup\{|f(p) - f(q)|;\ \|[D, \pi(f)\| \leq 1\}, \tag{58}$$

where $\pi(f)$ is the multiplication by f operator. The triple $(C^\infty(S^1), L^2(S^1), D)$ is an example of a *spectral triple* and (58) is a prototype of a very general *distance formula* of Connes that recovers the distance on a Riemannian spin^c manifold from its Dirac operator (cf. formula (57) in this section and the last chapter of [21]).

Appendix A

Compact operators, Fredholm operators, and abstract index theory

The theory of operators on Hilbert space is essential for noncommutative geometry. Operator theory is the backbone of von Neumann and C^*-algebras and these are natural playgrounds for noncommutative measure theory and topology. We saw, for example, that K-homology has a natural formulation in operator theoretic terms using compact and Fredholm operators and it is this formulation that lends itself to generalization to the noncommutative setup. Similarly, the more refined aspects of noncommutative geometry, like noncommutative metric and Riemannian geometry, can only be formulated through spectral invariants of operators on Hilbert space.

We assume that the reader is familiar with concepts of Hilbert space, bounded operators on Hilbert space, and basic spectral theory as can be found in the first chapters of, e.g. [55], [63], [35]. A good reference for ideals of compact operators is [65]. For basic Fredholm theory and abstract index theory we recommend [35], [55]. In this section H will always stand for a Hilbert space over the complex numbers and $\mathcal{L}(H)$ for the algebra of bounded linear operators on H. The adjoint of an operator T shall be denoted by T^*.

Our first task in this section is to introduce several classes of ideals in $\mathcal{L}(H)$, most notably ideals of compact operators and the Schatten ideals. Let $\mathcal{F}(H)$ denote the set of *finite rank* operators on H, i.e., operators whose range is finite dimensional. $\mathcal{F}(H)$ is clearly a two-sided $*$-ideal in $\mathcal{L}(H)$ and in fact it is easy to show that it is the smallest proper two-sided ideal in $\mathcal{L}(H)$.

Let

$$\mathcal{K}(H) := \overline{\mathcal{F}(H)}$$

be the norm closure of $\mathcal{F}(H)$. It is clearly a norm closed two-sided $*$-ideal in $\mathcal{L}(H)$. An operator T is called *compact* if $T \in \mathcal{K}(H)$. Let H_1 denote the closed unit ball of H. It can be shown that an operator $T \in \mathcal{L}(H)$ is compact if and only if the norm closure $\overline{T(H_1)}$ is a compact subset of H in norm topology. It follows that the range of a compact operator can never contain a closed infinite dimensional subspace. The spectrum of a compact operator is a countable subset of $\mathbb{C}$ with 0 as its only possible limit point. Any nonzero point in the spectrum is an eigenvalue whose corresponding eigenspace is finite dimensional. For a compact operator T, let

$$\mu_1(T) \geq \mu_2(T) \geq \mu_3(T) \geq \cdots$$

denote the sequence of *singular values* of T. By definition, $\mu_n(T)$ is the n-th eigenvalue of $|T| := (T^*T)^{\frac{1}{2}}$, the absolute value of T,

It can be shown that if H is separable and infinite dimensional, which is the case in almost all examples, then $\mathcal{K}(H)$ is the unique proper and closed two-sided ideal of $\mathcal{L}(H)$. In this case it is also the largest proper two-sided ideal of $\mathcal{L}(H)$. Thus for any other two sided *operator ideal* $\mathcal{J}$ we have

$$\mathcal{F}(H) \subset \mathcal{J} \subset \mathcal{K}(H).$$

An interesting point of view, advocated by Connes and of fundamental importance for noncommutative geometry [21], is that compact operators are the true counterparts of infinitesimals in noncommutative geometry. If we regard $\mathcal{L}(H)$ as a replacement for $\mathbb{C}$ in noncommutative geometry (as in going from *c-numbers* to *q-numbers* in quantum mechanics), then compact operators should be regarded as infinitesimals. Classically, an infinitesimal is a 'number' whose absolute value is less than any positive number! The following lemma shows that the norm of a compact operator can be made as small as we wish, provided we stay away from a finite dimensional subspace:

Lemma A.1. *Let T be a compact operator. For any $\varepsilon > 0$ there is a finite dimensional subspace $V \subset H$ such that $\|PTP\| < \varepsilon$, where P is the orthogonal projection onto the orthogonal complement of V.*

The first thorough study of the ideal structure of $\mathcal{L}(H)$ was done by Calkin [13]. Among the two-sided ideals of $\mathcal{L}(H)$, and perhaps the most important ones for noncommutative geometry, are the *Schatten ideals*, and ideals related to the Dixmier trace [21]. Let us recall the definition of the former class of ideals next.

A compact operator $T \in \mathcal{K}(H)$ is called a *trace class* operator if

$$\sum_{n=1}^{\infty} \mu_n(T) < \infty.$$

Let e_n, $n \geq 1$, be an orthonormal basis of H. It is easy to see that if T is trace class then

$$\mathrm{Tr}(T) := \sum_i \langle Te_i,\, e_i \rangle$$

is finite and is independent of the choice of basis. We denote the set of trace class operators by $\mathcal{L}^1(H)$. It is a two sided $*$-ideal in $\mathcal{L}(H)$. Using the definition of the trace Tr, it is easy to check that if A and B are both trace class, then

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA). \tag{A.1}$$

What is much less obvious though, and that is what we actually used in Chapter 4, is that if both AB and BA are trace class then (A.1) still holds. A proof of this can be given using *Lidski's theorem*. This theorem is one of the hardest facts to establish about trace class operators (cf. [65] for a proof).

Theorem A.1 (Lidski's theorem)**.** *If A is a trace class operator then*

$$\mathrm{Tr}(A) = \sum_{i}^{\infty} \lambda_i,$$

where the summation is over the set of eigenvalues of A.

Now since for any two operators A and B, AB and BA have the same spectrum (and spectral multiplicity) except for 0 (cf. Exercise **??**) we obtain the

Corollary A.1. *Assume A and B are bounded operators such that AB and BA are both trace class. Then* (A.1) *holds.*

Next we define the class of *Schatten-p* ideals for $p \in [1, \infty)$ by

$$\mathcal{L}^p(H) := \{T \in \mathcal{L}(H);\ |T|^p \in \mathcal{L}^1(H)\}.$$

Thus $T \in \mathcal{L}^p(H)$ if and only if

$$\sum_{n=1}^{\infty} \mu_n(T)^p < \infty.$$

It is clear that if $p \leq q$ then $\mathcal{L}^p(H) \subset \mathcal{L}^q(H)$. The Schatten p-norm is defined by

$$\|T\|_p^p = \sum_{n=1}^{\infty} \mu_n(T)^p.$$

Proposition A.1. 1) *$\mathcal{L}^p(H)$ is a two-sided ideal of $\mathcal{L}(H)$.*

2) (Hölder inequality) *Let $p, q, r \in [1, \infty]$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For any $S \in \mathcal{L}^p(H)$ and $T \in \mathcal{L}^q(H)$, we have $ST \in \mathcal{L}^r(H)$ and*

$$\|ST\|_r \leq \|S\|_p \|T\|_q.$$

In particular if $A_i \in \mathcal{L}^n(H)$ for $i = 1, 2, \ldots, n$, then their product $A_1 A_2 \ldots A_n$ is in $\mathcal{L}^1(H)$.

Example A.1. 1. Let us fix an orthonormal basis e_n, $n = 0, 1, 2, \dots$, in H. The diagonal operator defined by $Te_n = \lambda_n e_n$, $n \geq 0$ is compact if and only if $\lambda_n \to 0$ as $n \to \infty$. It is in $\mathcal{L}^p(H)$ if and only if $\sum_i |\lambda_i|^p < \infty$. By the spectral theorem for compact operators, every selfadjoint compact operator is unitarily equivalent to a diagonal operator as above.

2. Integral operators with L^2 kernels provide typical examples of operators in $\mathcal{L}^2(H)$, the class of *Hilbert–Schmidt operators*. Let K be a complex-valued square integrable function on $X \times X$ where (X, μ) is a measure space. Then the operator T_K on $L^2(X, \mu)$ defined by

$$(T_K f)(x) = \int_X K(x, y) f(y)\, d\mu$$

is a Hilbert–Schmidt (in particular compact) operator, with

$$\|T_K\|_2^2 = \|K\|_2^2 = \int_X \int_X |K(x, y)|^2\, dxdy.$$

Under suitable conditions, e.g. when X is compact Hausdorff and the kernel K is continuous, T_K is a trace class operator and

$$\mathrm{Tr}(T_K) = \int_X K(x, x)\, dx.$$

In the remainder of this section we shall recall some basic definitions and facts about Fredholm operators and index. A bounded linear operator $T\colon H_1 \to H_2$ between two Hilbert spaces is called a *Fredholm operator* if its kernel and cokernel are both finite dimensional:

$$\dim \ker(T) < \infty, \quad \dim \operatorname{coker}(T) < \infty.$$

The *index* of a Fredholm operator is the integer

$$\begin{aligned} \operatorname{index}(T) &:= \dim \ker(T) - \dim \operatorname{coker}(T) \\ &= \dim \ker(T) - \dim \ker(T^*). \end{aligned}$$

We list some of the standard properties of Fredholm operators and the index that are frequently used in noncommutative geometry:

1. (Atkinson's theorem) A bounded operator $T\colon H_1 \to H_2$ is Fredholm if and only if it is invertible modulo compact operators, that is, if there exists an operator $S\colon H_2 \to H_1$ such that $1 - ST$ and $1 - TS$ are compact operators on H_1 and H_2 respectively. S is called a *parametrix* for T. It can also be shown that T is Fredholm if and only if it is invertible module finite rank operators.

Let $\mathcal{C} := \mathcal{L}(H)/\mathcal{K}(H)$ denote the *Calkin algebra* and $\pi\colon \mathcal{L}(H) \to \mathcal{C}$ be the quotient map. (By general C^*-algebra theory, a quotient of a C^*-algebra by a closed two sided $*$-ideal is a C^*-algebra in a natural way). Thus Atkinson's theorem can be reformulated as saying that an operator T is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra. This for example immediately implies that Fredholm operators form an open subset of $\mathcal{L}(H)$ which is invariant under compact perturbations.

2. If T_1 and T_2 are Fredholm operators then T_1T_2 is also a Fredholm operator and

$$\operatorname{index}(T_1T_2) = \operatorname{index}(T_1) + \operatorname{index}(T_2).$$

3. The Fredholm index is stable under compact perturbations: if K is a compact operator and T is Fredholm, then $T + K$ is Fredholm and

$$\operatorname{index}(T + K) = \operatorname{index}(T).$$

4. The Fredholm index is a *homotopy invariant*: if T_t, $t \in [0, 1]$ is a norm continuous family of Fredholm operators then

$$\operatorname{index}(T_0) = \operatorname{index}(T_1).$$

It is this homotopy invariance, or continuity, of the index that makes it computable and extremely useful. Note that $\dim\ker T_t$ can have jump discontinuities.

5. Let $\mathrm{Fred}(H)$ denote the set of Fredholm operators on a separable infinite dimensional Hilbert space. It is an open subset of $\mathcal{L}(H)$ and the *index map*

$$\operatorname{index}\colon \mathrm{Fred}(H) \to \mathbb{Z}$$

induces a one-to-one correspondence between the connected components of $\mathrm{Fred}(H)$ and $\mathbb{Z}$.

6. $\mathrm{Fred}(H)$ is a *classifying space* for K-theory. More precisely, by a theorem of Atiyah and Jänich, for any compact Hausdorff space X, we have a canonical isomorphism of abelian groups

$$K^0(X) \simeq [X, \mathrm{Fred}(H)],$$

where $[X, \mathrm{Fred}(H)]$ is the set of homotopy classes of norm continuous maps from $X \to \mathrm{Fred}(H)$. Thus continuous families of Fredholm operators on X, up to homotopy, gives the K-theory of X.

7. (Calderón's formula [12]) Let $P\colon H_1 \to H_2$ be a Fredholm operator and let $Q\colon H_2 \to H_1$ be a parametrix for P. Assume that for some positive integer n, $(1 - PQ)^n$ and $(1 - QP)^n$ are both trace class operators. Then we have

$$\operatorname{index}(P) = \operatorname{Tr}(1 - QP)^n - \operatorname{Tr}(1 - PQ)^n. \tag{A.2}$$

Here is an alternative formulation of the above result. Let $H = H_1 \oplus H_2$. It is a *super Hilbert space* with even and odd parts given by H_1 and H_2. Let $F = \left(\begin{smallmatrix} 0 & Q \\ P & 0 \end{smallmatrix}\right)$ and γ be the corresponding grading operator. Then we have

$$\operatorname{index}(P) = \operatorname{Tr}_s(1 - F^2)^n, \tag{A.3}$$

where Tr_s is the *supertrace* of trace class operators, defined by $\operatorname{Tr}_s(X) = \operatorname{Tr}(\gamma X)$.

Example A.2. We give a few examples of Fredholm operators and Fredholm index.

1. Any operator $T: H_1 \to H_2$ where both H_1 and H_2 are finite dimensional is Fredholm. Its index is independent of T and is given by

$$\text{index}\,(T) = \dim(H_1) - \dim(H_2).$$

If only one of H_1 or H_2 is finite dimensional then no T can be Fredholm. This shows that the class of Fredholm operators and index is a purely infinite dimensional phenomenon and are only interesting when both H_1 and H_2 are infinite dimensional.

2. Let us fix an orthonormal basis e_n, $n = 0, 1, 2, \ldots$, for H. The *unilateral shift* operator is defined by

$$T(e_i) = e_{i+1}, \quad i \geq 0.$$

It is easy to see that T is injective and its range is the closed subspace spanned by $e_i, i \geq 1$. Thus T is a Fredholm operator with $\text{index}(T) = -1$. Its adjoint T^* (the backward shift) has index $+1$. Their powers T^m and T^{*m} are m-step forward and backward shifts, respectively, with $\text{index}(T^m) = m$ and $\text{index}(T^{*m}) = -m$.

3. We saw in Section 0.19 that for any odd Fredholm modules (H, F) over an algebra A and an invertible element $U \in A$ the operator $PUP: PH \to PH$ is a Fredholm operator, where $P = \frac{1+F}{2}$ is the projection onto the 1-eigenspace of F. Similarly for an even Fredholm module (H, F, γ) over A and an idempotent $e \in A$, the operator $F_e^+ : (eFe)^+ : e^+H_+ \to e^-H_-$ is Fredholm.

4. Elliptic differential operators acting on smooth sections of vector bundles over closed manifolds define Fredholm operators on the corresponding Sobolev spaces of sections. Computing the index of such Fredholm operators is what the index theorem of Atiyah–Singer achieves. Let M be a smooth manifold and let E and F be smooth complex vector bundles on M. Let

$$D: C^\infty(E) \to C^\infty(F)$$

be a liner differential operator. This means that D is a $\mathbb{C}$-linear map which is locally expressible by an $m \times n$ matrix of differential operators. This matrix of course depends on the choice of local coordinates on M and local frames for E and F. The *principal symbol* of D is defined by replacing differentiation by covectors in the leading order terms D. The resulting 'matrix-valued function on the cotangent bundle'

$$\sigma_D \in C^\infty(\text{Hom}(\pi^*E, \pi^*F))$$

can be shown to be invariantly defined. Here $\pi: T^*M \to M$ is the natural projection map of the cotangent bundle. A differential operator D is called elliptic if for all $x \in M$ and all nonzero $\xi \in T_x^*M$, the principal symbol $\sigma_D(x, \xi)$ is an invertible matrix.

Let $W^s(E)$ denote the Sobolev space of sections of E (roughly speaking, it consists of sections whose 'derivatives of order s' are square integrable). The main results of the theory of linear elliptic PDE's show that for each $s \in \mathbb{R}$, D has a unique extension to a bounded and Fredholm operator $D\colon W^s(E) \to W^{s-n}(F)$ between Sobolev spaces (n is the order of the differential operator D). Moreover the Fredholm index of D is independent of s and coincides with the index defined using smooth sections.

Remark 5. Even in a purely algebraic context, the notion of *Fredholm operator* makes sense and defines an interesting class of linear operators. A linear map : $T : V_1 \to V_2$ is called Fredholm if its kernel and cokernel are both finite dimensional. In that case we can define the index of T as the difference

$$index(T) = dim(kerT) - dim(cokerT).$$

It is easy to see that this concept is interesting only if V_1 and V_2 are both infinite dimensional. The stongest results are obtained when T is a *bounded* linear operator between Hilbert spaces.

Exercise A.1. Any invertible operator is clearly Fredholm and its index is zero. Thus any compact perturbation of a an invertible operator is Fredholm and its index is zero. Is it true that any Fredholm operator with zero index is a compact perturbation of an invertible operator?

Exercise A.2. Prove Calderón's formula (A.2) for $n = 1$.

Exercise A.3. Formula (A.3) relates the Fredholm index with the operator trace. Here is a similar formula. Let H be a $\mathbb{Z}_2$-graded Hilbert space and let D be an unbounded *odd* selfadjoint operator on H such that e^{-tD^2} is a trace class operator for all $t > 0$. Show that $\mathrm{index}(D) := \dim\ker(D^+) - \dim\ker(D^-)$ is well defined and is given by the *McKean–Singer formula*

$$\mathrm{index}(D) = \mathrm{Tr}_s(e^{-tD^2}) \quad \text{for all } t > 0.$$

Appendix B

K-theory

We start by briefly recalling the definitions of the functors K_0 and K_1. Let A be a unital algebra and let $\mathcal{P}(A)$ denote the set of isomorphism classes of finitely generated projective right A-modules. Under the operation of direct sum, $\mathcal{P}(A)$ is an abelian monoid. The group $K_0(A)$ is, by definition, the *Grothendieck group* of the monoid $\mathcal{P}(A)$ in the sense that there is a universal additive map $\mathcal{P}(A) \to K_0(A)$. Thus elements of $K_0(A)$ can be written as $[P]-[Q]$ for P, $Q \in \mathcal{P}(A)$, with $[P]-[Q] = [P']-[Q']$ if and only if there is an $R \in \mathcal{P}(A)$ such that $P \oplus Q' \oplus R \simeq P' \oplus Q \oplus R$.

There is an alternative description of $K_0(A)$ in terms of idempotents in matrix algebras over A that is often convenient. An idempotent $e \in M_n(A)$ defines a right A-module map

$$e \colon A^n \to A^n$$

by left multiplication by e. Let $P_e = eA^n$ be the image of e. The relation

$$A^n = eA^n \oplus (1-e)A^n$$

shows that P_e is a finite projective right A-module. Different idempotents may define isomorphic modules. This happens, for example, if e and f are *equivalent idempotents* (sometimes called *similar*) in the sense that

$$e = ufu^{-1}$$

for some invertible $u \in \mathrm{GL}(n, A)$. Let $M(A) = \bigcup M_n(A)$ be the direct limit of the matrix algebras $M_n(A)$ under the embeddings $M_n(A) \to M_{n+1}(A)$ defined by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Similarly let $\mathrm{GL}(A)$ be the direct limit of the groups $\mathrm{GL}(n, A)$. It acts on $M(A)$ by conjugation.

Definition B.0.1. Two idempotents $e \in M_k(A)$ and $f \in M_l(A)$ are called *stably equivalent* if their images in $M(A)$ are equivalent under the action of $\mathrm{GL}(A)$.

The following is easy to prove and answers our original question.

Lemma B.0.1. *The projective modules P_e and P_f are isomorphic if and only if the idempotents e and f are stably equivalent.*

Let $\mathrm{Idem}(M(A))/\,\mathrm{GL}(A)$ denote the set of stable equivalence classes of idempotents over A. This is an abelian monoid under the operation

$$(e,\,f) \mapsto e \oplus f := \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

It is clear that any finite projective module is of the type P_e for some idempotent e. In fact writing $P \oplus Q \simeq A^n$, one can let e be the idempotent corresponding to the projection map $(p, q) \mapsto (p, 0)$. These observations prove the following lemma.

Lemma B.0.2. *For any unital ring A, the map $e \mapsto P_e$ defines an isomorphism of monoids*

$$\mathrm{Idem}(M(A))/\,\mathrm{GL}(A) \simeq \mathcal{P}(A).$$

Given an idempotent $e = (e_{ij}) \in M_n(A)$, its image under a homomorphism $f\colon A \to B$ is the idempotent $f_*(e) = (f(e_{ij}))$. This is our formula for $f_*\colon K_0(A) \to K_0(B)$ in the idempotent picture of K-theory. It turns $A \to K_0(A)$ into a functor from unital algebras to abelian groups.

For a unital Banach algebra A, $K_0(A)$ can be described in terms of connected components of the space of idempotents of $M(A)$ under its inductive limit topology (a subset $V \subset M(A)$ is open in the inductive limit topology if and only if $V \cap M_n(A)$ is open for all n). It is based on the following important observation: Let e and f be idempotents in a unital Banach algebra A and assume $\|e - f\| < 1/\|2e - 1\|$. Then $e \sim f$. In fact with

$$v = (2e - 1)(2f - 1) + 1 \tag{B.1}$$

and $u = \frac{1}{2}v$, we have $ueu^{-1} = f$. To see that u is invertible note that $\|u - 1\| < 1$. One consequence of this fact is that if e and f are in the same path component of the space of idempotents in A, then they are equivalent. As a result we have, for any Banach algebra A, an isomorphism of monoids

$$\mathcal{P}(A) \simeq \pi_0(\mathrm{Idem}(M(A))),$$

where π_0 is the functor of path components.

For C^*-algebras, instead of idempotents it suffices to consider only the *projections*. A projection is a self-adjoint idempotent ($p^2 = p = p^*$). The reason is that every idempotent in a C^*-algebra is similar to a projection [7]: let e be an idempotent and set $z = 1 + (e - e^*)(e^* - e)$. Then z is invertible and positive and one shows that $p = ee^*z^{-1}$ is a projection and is similar to e. In fact, it can be shown that the set of projections of a C^*-algebra is a retraction of its set of idempotents. Let $\mathrm{Proj}(M(A))$ denote the space of projections in $M(A)$. We have established isomorphisms of monoids

$$\mathcal{P}(A) \simeq \pi_0(\mathrm{Idem}(M(A))) \simeq \pi_0(\mathrm{Proj}(M(A)))$$

which reflects the coincidence of stable equivalence, Murray–von Neumann equivalence, and homotopy equivalence in Proj $(M(A))$.

Starting with K_1, algebraic and topological K-theory begin to differ from each other. We shall briefly indicate the definition of algebraic K_1, and the necessary modification needed for topological K^1. Let A be a unital algebra. The algebraic K_1 of A is defined as the *abelianization of* the group $\mathrm{GL}(A)$:

$$K_1^{\mathrm{alg}}(A) := \mathrm{GL}(A)/[\mathrm{GL}(A),\, \mathrm{GL}(A)],$$

where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator subgroup. Applied to $A = C(X)$, this definition does not reproduce the topological $K^1(X)$. For example for $A = \mathbb{C} = C(\mathrm{pt})$ we have $K_1^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{C}^\times$ where the isomorphism is induced by the determinant map

$$\det\colon\ \mathrm{GL}(\mathbb{C}) \to \mathbb{C}^\times,$$

while $K^1(\mathrm{pt}) = 0$. It turns out that, to obtain the right result, one should divide $\mathrm{GL}(A)$ by a bigger subgroup, i.e., by the *closure* of its commutator subgroup. This works for all Banach algebras and will give the right definition of topological K_1. A better approach however is to define the higher K groups in terms of K_0 and the *suspension functor* [7].

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